# The Complete Book of CHISANBOP™

## Original Finger Calculation Method

Sung Jin Pai & Hang Young Pai

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# CHISANBOP™

## ORIGINAL FINGER CALCULATION METHOD


Created by
SUNG JIN PAI and HANG YOUNG PAI

Written by
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Edited by
JOHN LEONARD



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# PREFACE

In the very short time since I immigrated to the United States, I have seen my Chisanbop™ Method spread throughout the American classrooms. It has been the fulfillment of a dream that I have had for many years. We, in Korea, have become quite used to having the latest advances of American education introduced into our educational system. It was time, I felt, that we paid back part of our debt.

It is with particular pleasure that I present the section on subtraction, especially because it gives me an opportunity to explain the rules for HYPAI Trading. While Chisanbop™ was first created by my father, Sung Jin Pai, the trading method was one of my recent developments. It has met with almost universal acclaim by teachers, who have been presenting it as a method for dealing with borrowing.

Chisanbop™ has flourished only because of the generous contributions of hundreds of people. To them I will always be grateful. Among those I feel honored to cite is first of all my father, Sung Jin Pai, who created the method over thirty years ago. His genius and hard work are most responsible for the success of the Chisanbop™ Method.

Important to the early dissemination of this methodology were the children whom I trained and who appeared on various national TV shows. Their personalities and skill at Chisanbop™ won millions of followers. Their competence made any adult commentary superfluous.

The administrators and teachers in the Mount Vernon School system blazed a trail soon to be followed in every state of the Union, Canada, Mexico, Europe, and Australia. These early pioneers will always deserve a special honor for their bravery in defying tradition and encouraging children to use their fingers.

I am grateful to my editor, John A. Leonard, who so ably interpreted for me. His insights into the conventional methods provided me with several opportunities for presenting easier and more consistent procedures.

I would also like to take this opportunity to thank the more than 3,000 teachers who have trained in Chisanbop™ through the 30-hour workshop during 1978–1979, and even more so, their trainers and instructors who have helped spread the Chisanbop™ message throughout the U.S. and Canada. I am especially grateful to those who have travelled to spread the word in Europe, Australia and Mexico.

The publication of this book owes a very great debt to Mr. Lawrence Jackel and Mr. Anthony Quaglia. Their early enthusiasm and long-term commitment of energy and insight stood as pillars of support as I completed this manuscript.

While deeply indebted to all the people I have mentioned and many others far too numerous to mention, there are two, without whom everything would have been impossible—my wife and inspiration, Hee Ja Park Pai, and Dr. Robert L. Leslie, founder and president of the Typophiles, my sponsor, friend, and mentor.

Hang Young Pai

# INTRODUCTION

Our nation has been repeatedly astounded watching 7-year-old children perform complicated problems and get accurate answers quicker than adults using hand calculators. These children trained by the author, using their fingers, outperformed Johnny Carson on the Tonight Show and appeared on every other national interview show. Their speed and accuracy excited Americans. Their self-confidence and freedom from machines made many an older viewer envious. They had learned a system that you are about to study, an easy scientific system of using fingers to perform arithmetical calculations called Chisanbop™.

Chisanbop™ is the sound of three Korean words meaning: Finger Calculation Method. Chisanbop™ was created in Korea over 30 years ago by Sung Jin Pai. His son, Hang Young Pai, simplified and refined the system so that it could be taught to very young children. It was brought to this country a little more than three years ago by the son.

The most important reason why the Chisanbop™ Method has been so successful is that it brings an added sense to the learning of arithmetic. Abstract ideas become concrete through the use of the fingers. All of us know that babies learn through all of their senses. Watch how a very young child will place a new object in his mouth, take it out and stare at it; bend or fold it if possible, bang it against the crib to hear it, and even throw it on the floor to see if it breaks. Unfortunately, most teachers never use the sense of touch when teaching arithmetic. But now, they have a method that uses this tactile sense. Children can now learn arithmetic not only by seeing, hearing, and speaking, but also by feeling. The sense of touch not only provides another inlet into the brain, but it abbreviates the time required to learn. It brings a game-like atmosphere into the classroom and makes learning fun.

In the last three years, Chisanbop™ has met wide acceptance by educators. They recognize that Chisanbop™ is not a gimmick or another revolutionary "new" math but a return to basics. Chisanbop™ does not replace traditional mathematics, but *supplements* it and makes it easier.

There are three main features of Chisanbop™. 1) the fingers are used; accuracy is thus improved. The fact that students find a way to be always accurate enhances their self-confidence and makes learning painless. 2) Chisanbop™ introduces a fresh approach to the mathematical operations by using *easy consistent* rules. Since the rules are consistent, children have less to remember. 3) Presented here are sequenced series of problems that the students learn one step at a time. By carefully breaking down the matter to be learned and carefully presenting the material in a manner that conforms to the way students learn, learning becomes not a challenge, but an exciting adventure. Now, with the use of Chisanbop™, teachers are, for the first time in years, hearing requests for *more* homework.

Most adults remember being reprimanded for using their fingers to add. With America's children leading the way, we know that this restriction was wrong. As you progress through the 4 stages of Chisanbop™ you will come to a point where you will actually be feeling the answers on your fingers. Moreover, if you truly study and practice all 4 stages, you will find yourself calculating with greater speed and accuracy.

Chisanbop™ must be learned, stage-by-stage. Thus you must start with Addition, Stage 1. There are, however, certain limitations for very young or very slow students. Kindergarteners and 1st graders usually stay at stage 1. Second and third graders are capable of stage 2. Stage 3 is usually introduced to fourth and fifth graders while most students in the sixth or higher grades are capable of stage 4.

The subtraction section follows the same arrangement as the first part, breaking the operation of subtraction into 4 stages. An important word of warning must be made about the schedule under which Chisanbop™ is learned. At each stage, Chisanbop™ Subtraction must be delayed until after Chisanbop™ Addition is learned. Before attempting Subtraction Stage 1, every aspect of Addition Stage 1 must be thoroughly understood. Indeed, some teachers suggest proceeding to Addition Stage 2, before introducing Subtraction Stage 1.

## THE ORAL EXPRESSION OF NUMBERS

At a very young age, children begin to learn their numbers by hearing them and saying them. They hear and repeat: one, two, three, four, five, etc.". Children have no problems learning to say their numbers. The problems begin for many when they must learn to recognize the abstract written symbol and connect it with the spoken word. There must be consistency between the verbal and arabic number system. Our use of numbers is based on the decimal system—this means that we proceed in groups of 10. The first group of numbers is 0 through 9, the second group of numbers is 10 through 19, the third group of numbers is 20 through 29, etc. Although the oral and written number system of the numbers from 0 to 9 and the numbers from 20 to 29 etc., follow the decimal system, the oral numbers from 10 to 19 do not conform to the written numbers and are confusing to children—especially in the beginning.

For example, when the child says the number "27," he writes a "2" followed by a "7." However, when the child says the number "17," he must mentally reverse this process and write a "1" followed by a "7" even though he has first pronounced the seven. It is for this reason that youngsters so often have difficulty with the numbers from 10 through 19 and frequently write 91 for 19, 81 for 18, etc.

The way to correct this for younger children is to introduce the following system that replaces the verbal numbers from ten through nineteen as follows:

| | | | |
|---|---|---|---|
| ten | — | onety | (10) |
| eleven | — | onety-one | (11) |
| twelve | — | onety-two | (12) |
| thirteen | — | onety-three | (13) |
| fourteen | — | onety-four | (14) |
| fifteen | — | onety-five | (15) |
| sixteen | — | onety-six | (16) |
| seventeen | — | onety-seven | (17) |
| eighteen | — | onety-eight | (18) |
| nineteen | — | onety-nine | (19) |

This nomenclature is used because it is based on the decimal system. By saying "onety-five," for example, we have called out the tens' place (onety) first and the units' place (five) next. The child immediately knows that onety stands for one ten and five stands for five units. In this way the oral number and the written number are consistent, and upon hearing onety-five, the child automatically writes 15, without confusion.

Before we begin teaching Chisanbop ™, this oral expression of numbers must be firmly established. *Later, when this system has become internalized, the child can interchange it with the familiar terms, ten, eleven, twelve, etc.*

## INCREASING THE TACTILE SENSATION

With some students, even the introduction of the tactile sense, in this our multisensory approach to the teaching of arithmetic, is not enough. The tactile sense can be strengthened. If, with these very slow learners, you ask them to press their fingers on their lips or cheeks, they will have an increased tactile impression. This is especially true as the children learn to count up to ten. Once they have reached the level

of knowing and performing the two key manipulations of Stage 1, they can return to the single tactile sensation of pressing their fingers on a desk or table.

## HAND DIAGRAM SYMBOLS

There will be numerous illustrations of the proper finger positions throughout this book.

If a finger is shaded *black with an arrow pointing down* (↓) that finger should be pressed (fig. 1). If a finger is shaded black, but it has no arrow it means that finger was already pressed from a prior manipulation (fig. 2). If a finger has been pressed and now should be cleared, the finger is blank with an arrow pointing up (↑), (fig. 3). If a finger has no shading and has no arrow, it means the finger is off the table (fig. 4).

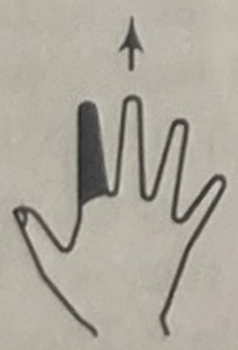
**Figure 1**

press 1st two fingers

**Figure 2**

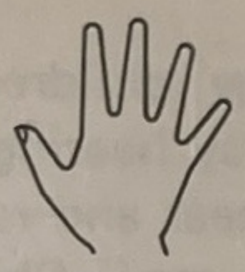
3 fingers pressed from prior manipulations

**Figure 3**



clear 2nd finger

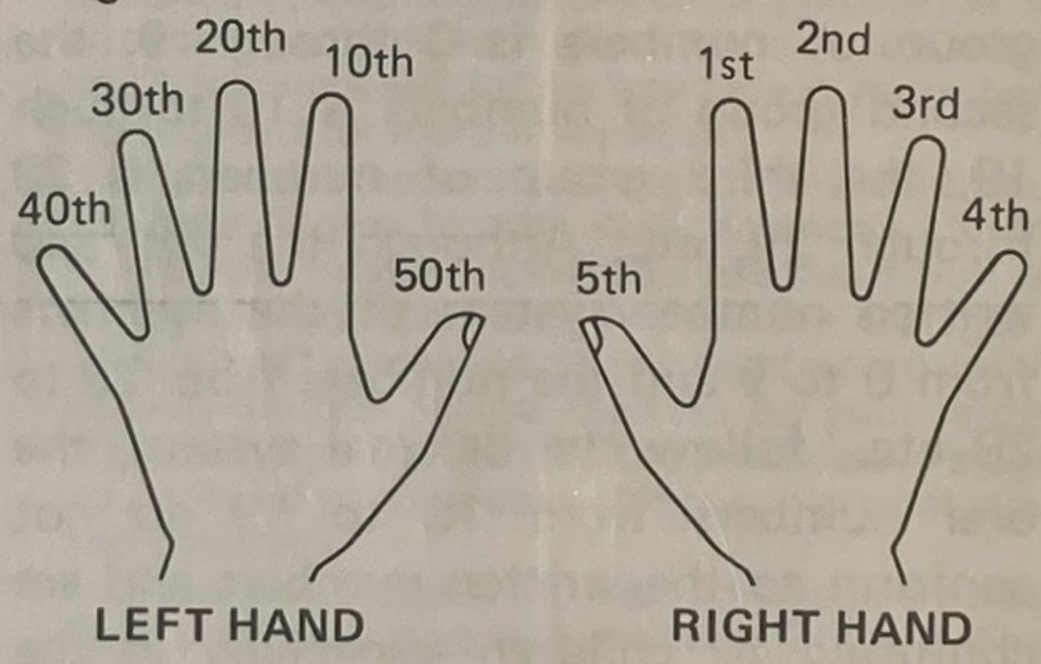
**Figure 4**



No fingers are pressed

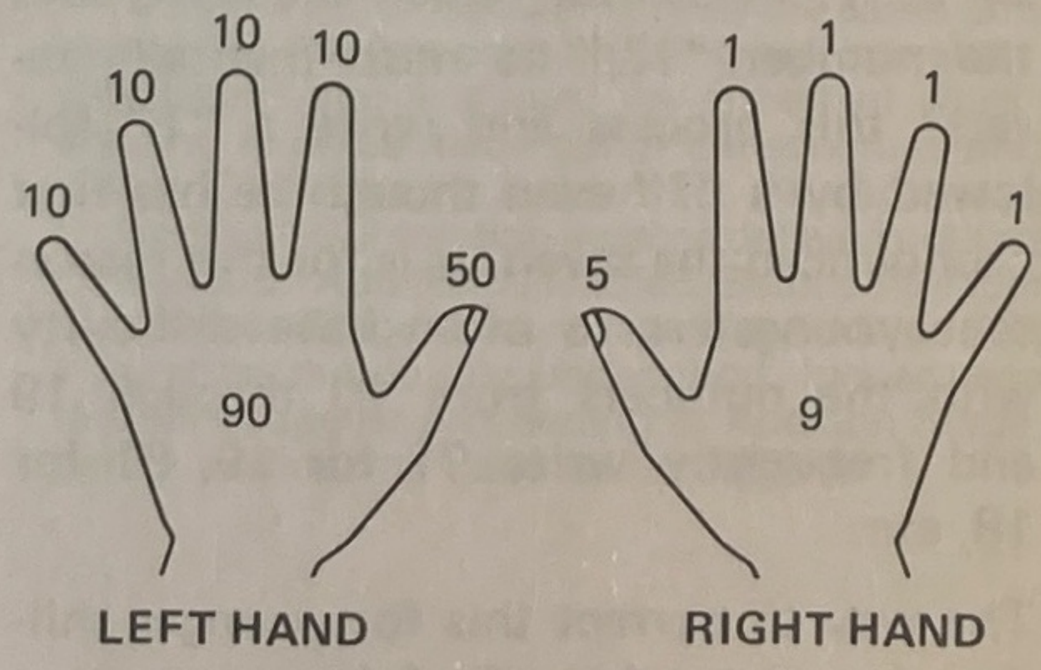
## NAMES FOR THE FINGERS

Chisanbop ™ uses the hands to accumulate various numbers on the fingers. Each finger has a specific value and there is a ten - to - one ratio between the left and right hand. First, we must name the fingers.



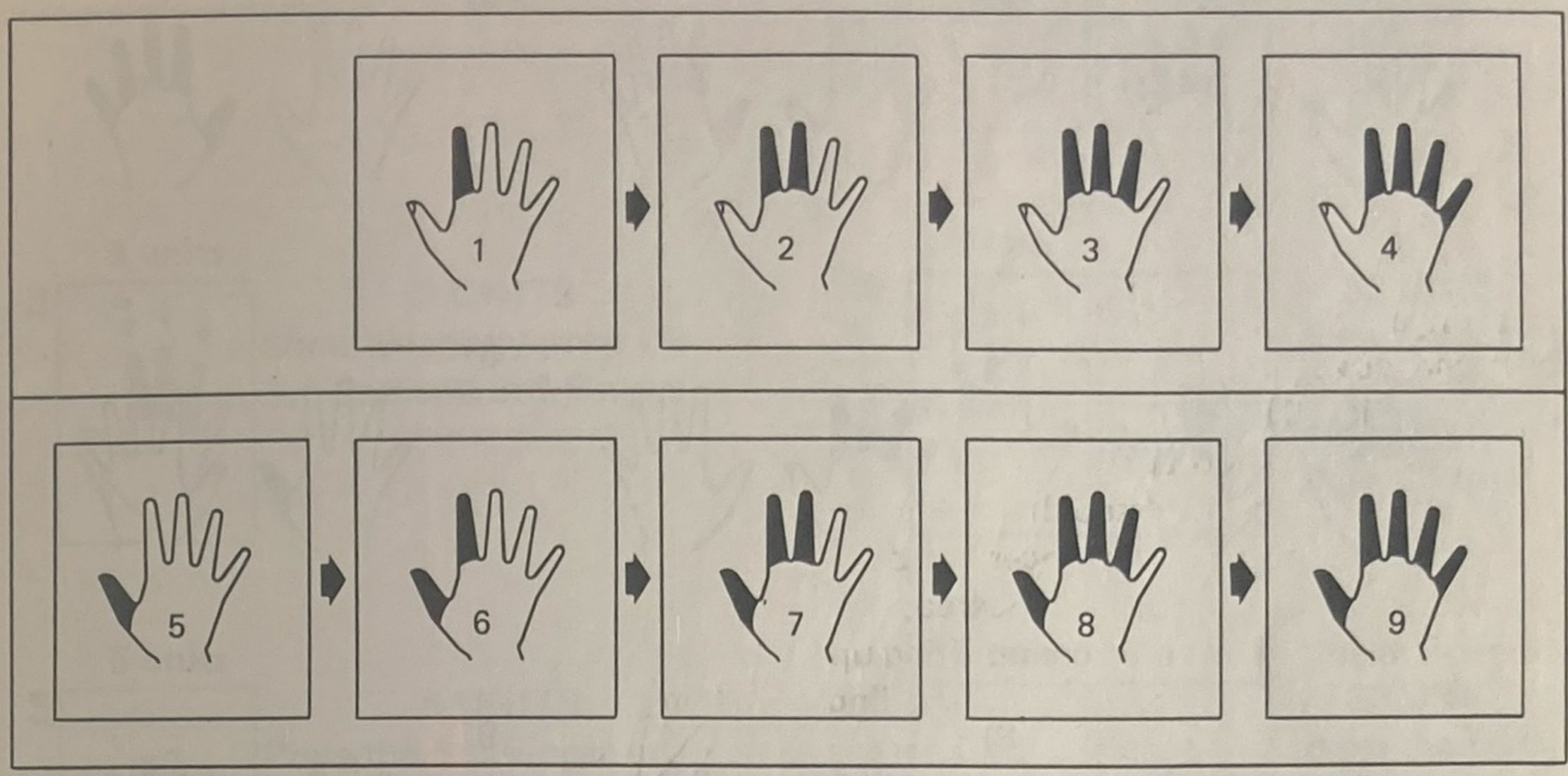

## VALUES OF THE FINGERS

Each finger has a specific value as shown below.

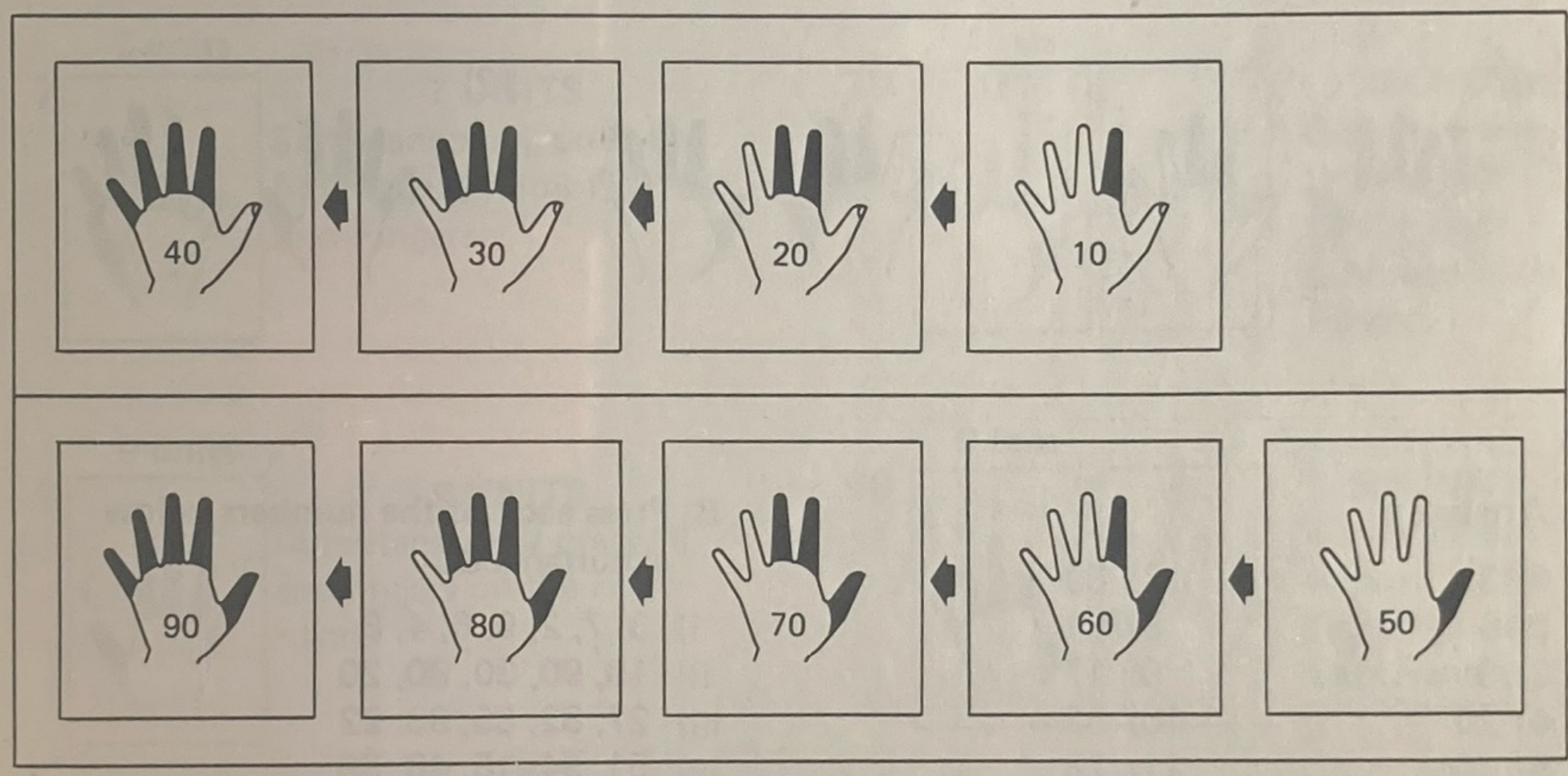


# HOW TO REPRESENT NUMBERS

## Right Hand

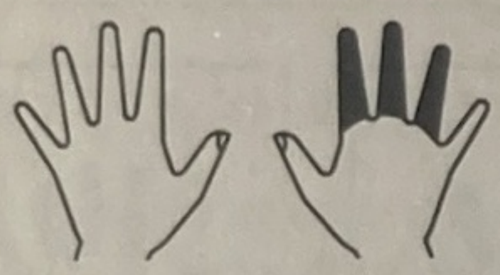


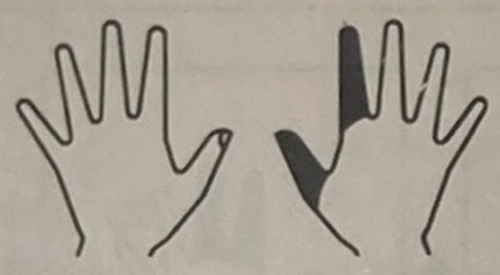

## Left Hand

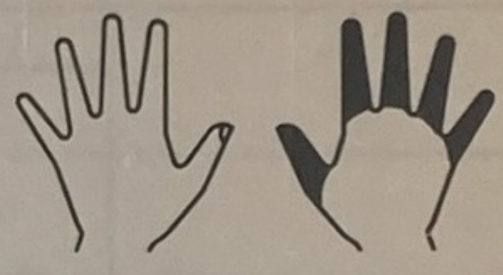


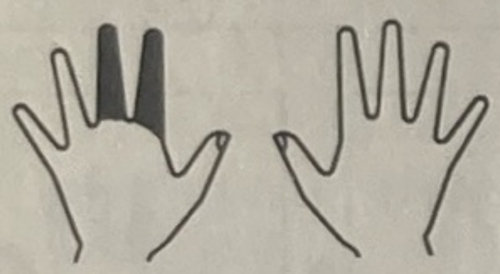
## EXERCISES

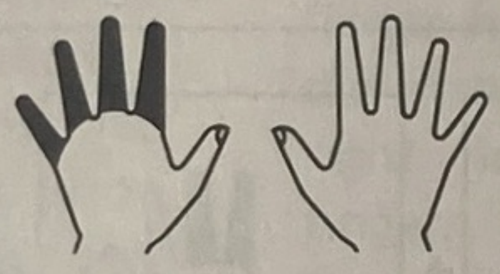
A) Read the values on the hands shown below.

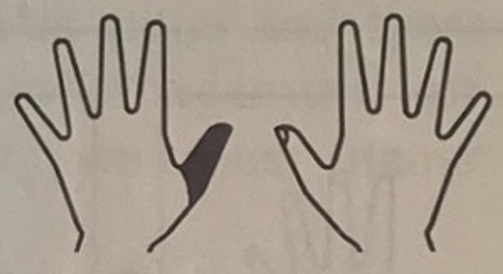
1) 

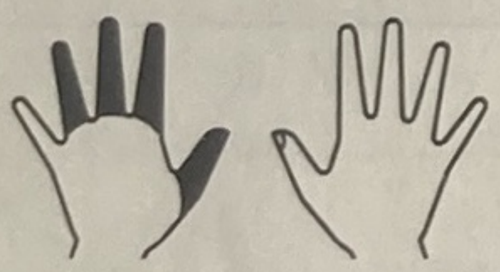
2) 

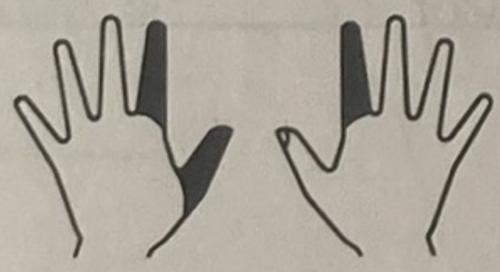
3) 

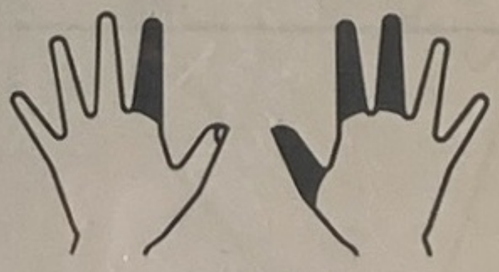
4) 

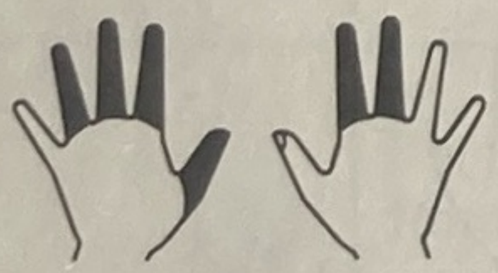
5) 

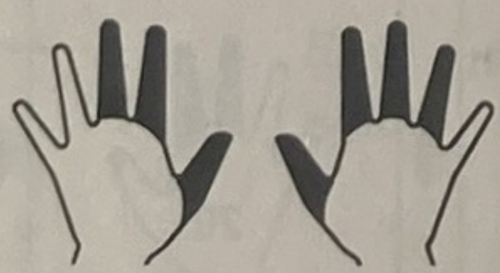
6) 

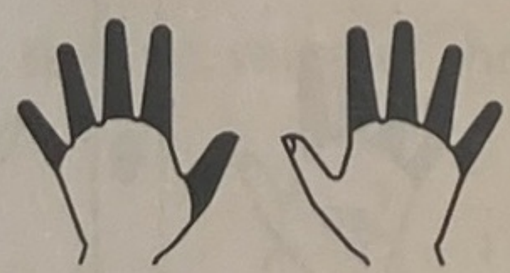
7) 

8) 

9) 

10) 

11) 

12) 

### Answers

1) 3
2) 6
3) 9
4) 20
5) 40
6) 50
7) 80
8) 61
9) 17
10) 82
11) 78
12) 94

B) Press each of the numbers below simultaneously

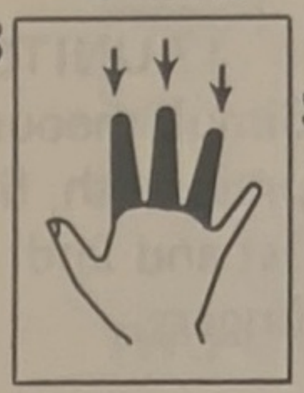
i) 3, 7, 2, 9, 6, 4, 8
ii) 10, 90, 30, 80, 20
iii) 27, 82, 65, 93, 22
iv) 51, 94, 16, 48, 39
v) 13, 95, 68, 32, 83

## SIMULTANEOUS PRESS

Manipulations such as the following should be practiced until they become completely automatic:

3 units

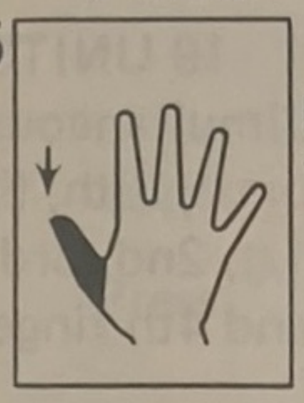
3



3 UNITS
Simultaneously press the 1st, 2nd and 3rd Fingers.

5 units

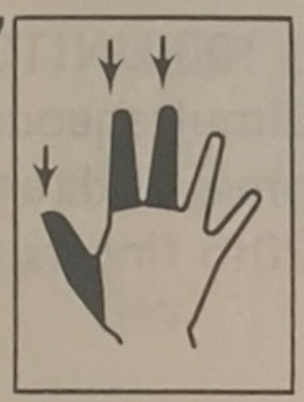
5



5 UNITS
Press the 5th Finger or Thumb.

7 units

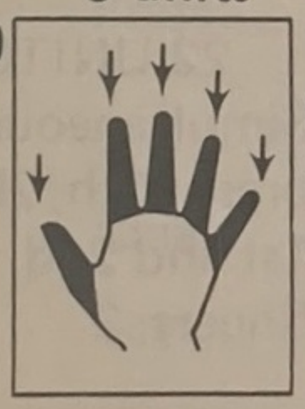
7



7 UNITS
Simultaneously press the 5th Finger (thumb), 1st and 2nd Fingers.

9 units

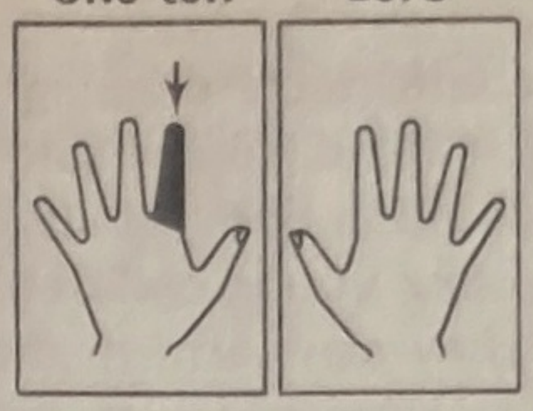
9



9 UNITS
Simultaneously press all the fingers on the right hand.

one ten zero

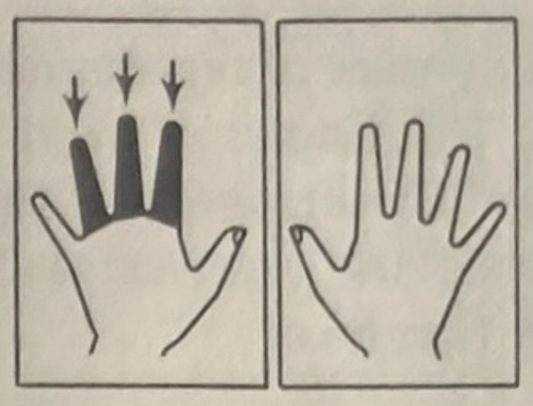
10



10 UNITS
Press the 10th Finger.

3 tens

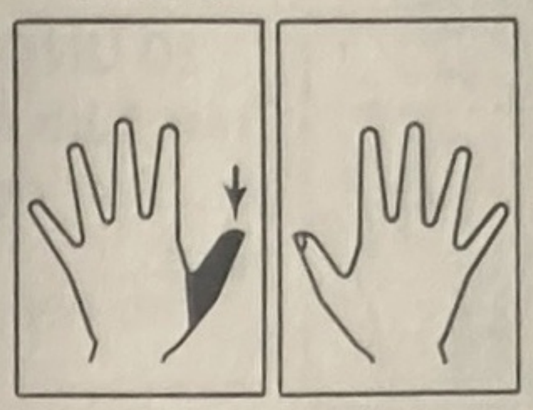
30



30 UNITS
Simultaneously press the 10th, 20th and 30th Fingers.

5 tens

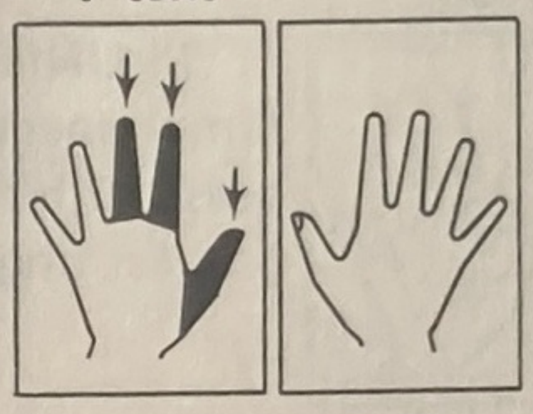
50



50 UNITS
Press the 50th Finger. (thumb)

7 tens

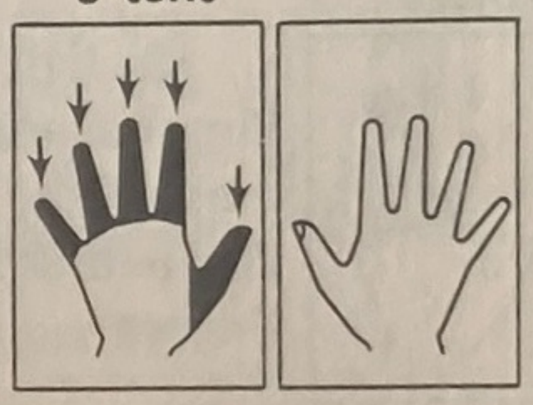
70



70 UNITS
Simultaneously press the 50th Finger, 10th and 20th Finger.

9 tens

90



90 UNITS
Simultaneously press all the fingers on the Left Hand.

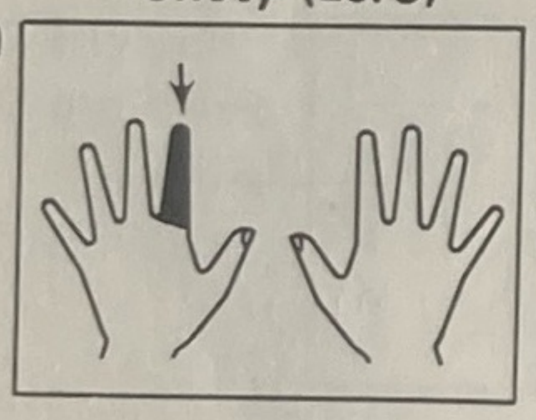
## Both Hands

Once you are comfortable pressing numbers on your left hand you should now begin to practice using both hands simultaneously. For very young children, it is sufficient that they can perform the manipulations up to twenty. Older children and adults should practice exercises up to the number 99.

Following are examples of representations of both hands. The fingers of each hand should be pressed simultaneously; the left hand may precede the right since this is the number you first hear.

**onety-(zero)**

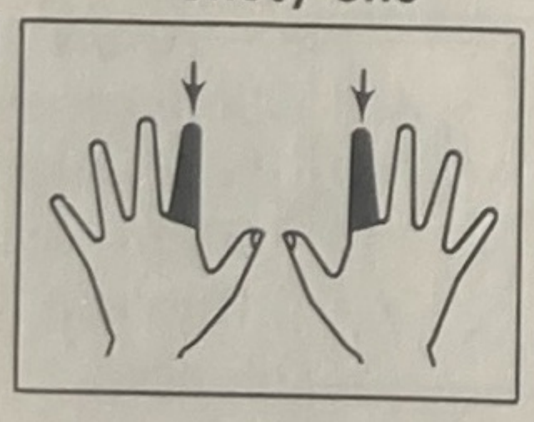
10



10 UNITS
Press 10th finger, clear right hand.

**onety-one**

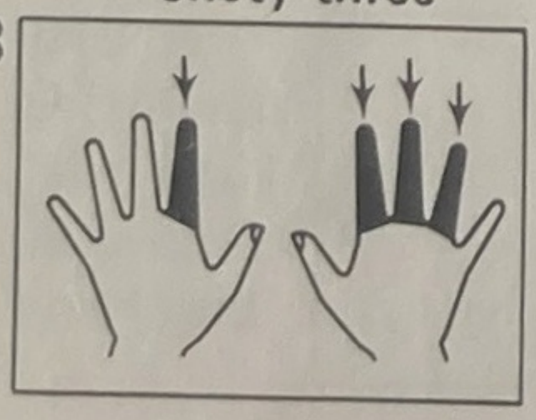
11



11 UNITS
Simultaneously press 10th finger and 1st finger.

**onety-three**

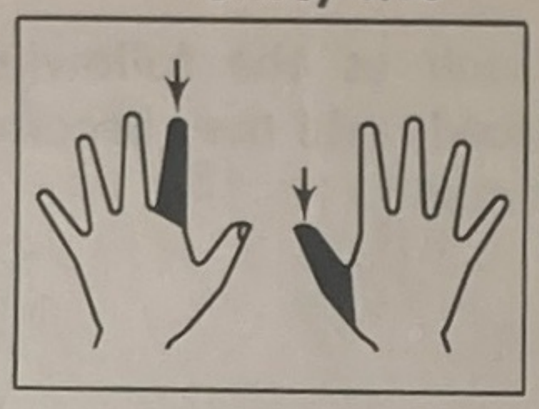
13



13 UNITS
Simultaneously press 10th, 1st, 2nd and 3rd fingers.

**onety-five**

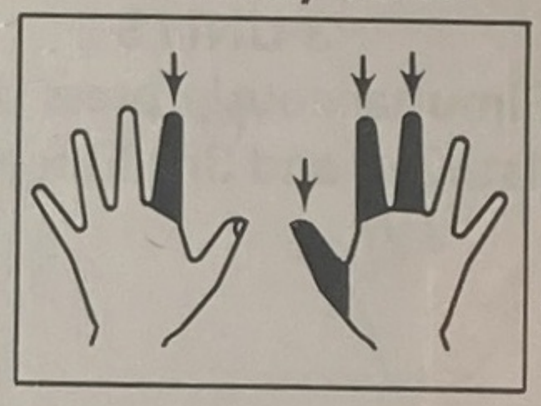
15



15 UNITS
Simultaneously press 10th and 5th fingers.

**onety-seven**

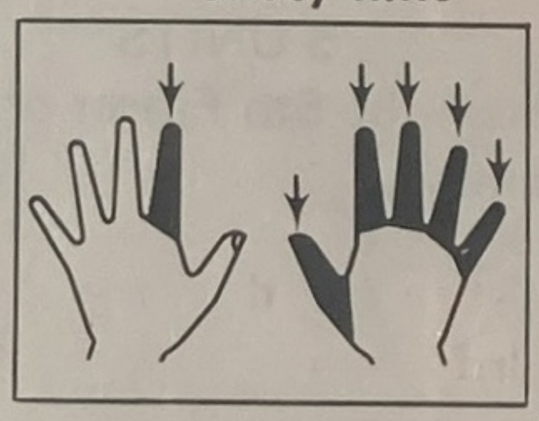
17



17 UNITS
Simultaneously press 10th, 5th, 1st and 2nd fingers.

**onety-nine**

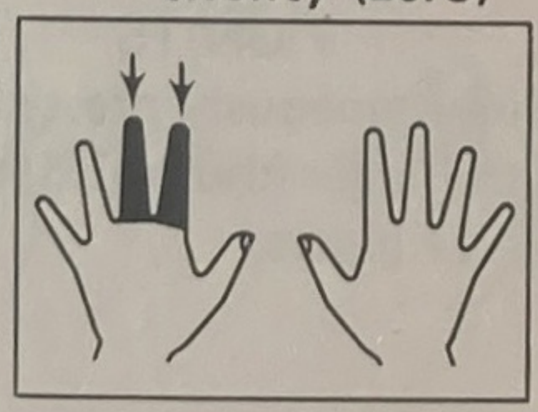
19



19 UNITS
Simultaneously press 10th, 5th, 1st, 2nd, 3rd, and 4th fingers.

**twenty-(zero)**

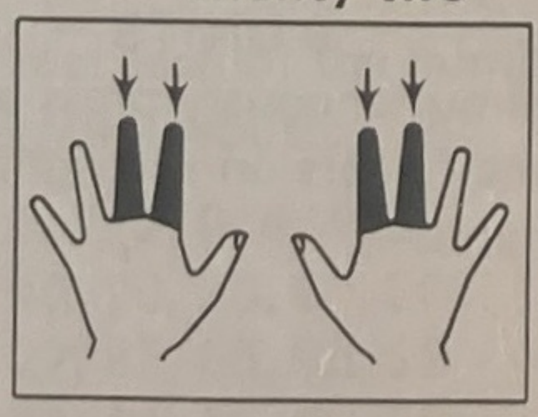
20



20 UNITS
Simultaneously press 10th and 20th fingers.

**twenty-two**

22



22 UNITS
Simultaneously press 10th, 20th; 1st and 2nd fingers.

## ADDITION STAGE 1

After pressing numbers simultaneously, you are now ready to begin adding. The key to stage 1 addition is to press each number, one finger at a time, and at the same time orally count out the number. The following pages illustrate how to press the first number in an addition problem.

## PRESSING ONE BY ONE UP TO 9

### 1 ONE UNIT

Press:* 1st finger.

Say: One.

### 2 TWO UNITS

Press:* 1st finger and then the 2nd finger.

Say: One, two.

### 3 THREE UNITS

Press:* 1st finger, then the 2nd finger, and then the 3rd finger.

Say: One, two, three.

### 4 FOUR UNITS

Press:* 1st finger, then the 2nd finger, then the 3rd finger, and then the 4th finger.

Say: One, two, three, four.

### 5 FIVE UNITS

Press:* 1st finger, then the 2nd finger, then the 3rd finger, then the 4th finger, and then the 5th finger.

Clear: 1st four fingers as the 5th finger is pressed.**

Say: One, two, three, four, five (clear 4 press 5).

*Press fingers one by one.

**Flip Manipulation

## 6 SIX UNITS

(a) Press:* 1st finger, then the 2nd finger, then the 3rd finger, then the 4th finger, then the 5th finger.

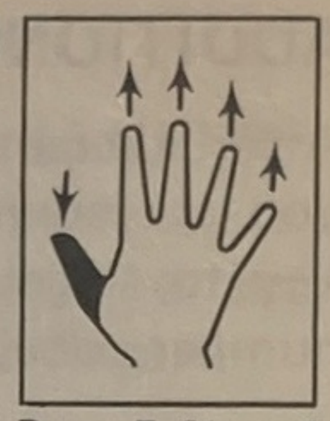
(a) Clear: 1st four fingers as the 5th finger is pressed.**

(b) Press:* Then the 1st finger again.

Say: (a) One, two, three, four, five (by pressing 5 and clearing 4).***

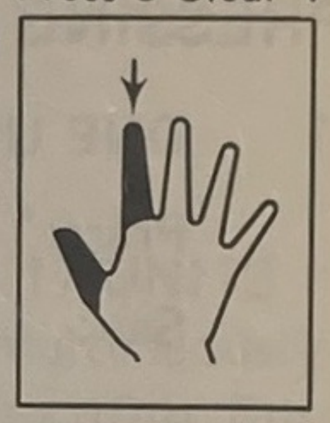
(b) six.

(a)



Press 5 Clear 4

(b)



## 7 SEVEN UNITS

(a) Press:* 1st finger, then the 2nd finger, then the 3rd finger, then the 4th finger, then the 5th finger.

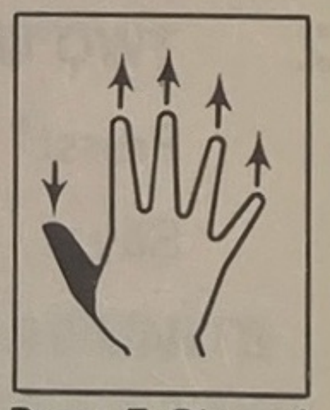
(a) Clear: 1st four fingers as the 5th finger is pressed.**

(b) Press:* Then the 1st finger again, and then the 2nd finger.

Say: (a) One, two, three, four, five (by pressing 5 and clearing 4).***

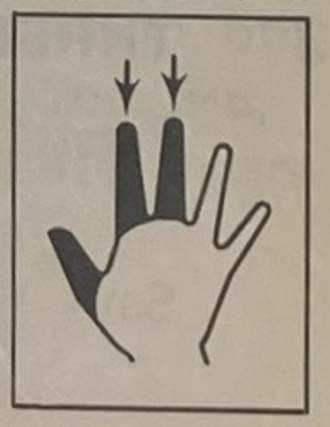
(b) Six, seven.

(a)



Press 5 Clear 4

(b)



## 8 EIGHT UNITS

(a) Press:* 1st finger, 2nd finger, 3rd finger, 4th finger, 5th finger.

(a) Clear: 1st four fingers as the 5th finger is pressed.**

(b) Press: 1st finger, 2nd finger, 3rd finger.

Say: (a) One, two, three, four, five (by pressing 5 and clearing 4).***

(b) Six, seven, eight.

(a)

(b)

*Press fingers one by one in order.

**Flip Manipulation

***The verbal instruction "by pressing 5 and clearing 4" can be eliminated once the Flip Manipulation is mastered.

## 9 NINE UNITS

(a) Press:* 1st finger, 2nd finger, 3rd finger, 4th finger, 5th finger.

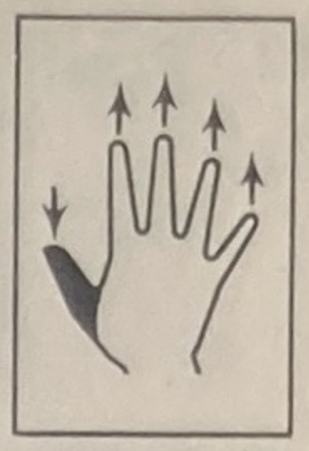
(a) Clear: 1st four fingers as the 5th finger is pressed.**

(b) Press: 1st finger, 2nd finger, 3rd finger, 4th finger.

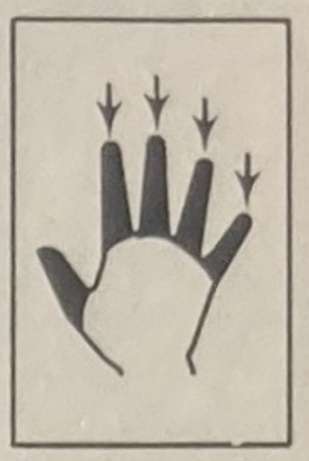
Say: (a) One, two, three, four, five (by pressing 5 and clearing 4).***

(b) Six, seven, eight, nine.

(a)



(b)



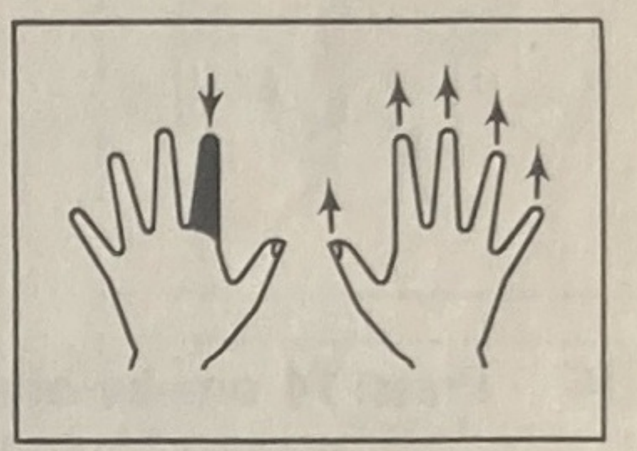
## PRESSING ONE BY ONE UP TO 20

After we establish the numbers from one to nine, we then see how to reach the number "onety" (ten). When no fingers on the right hand are available for pressing (that is, all of the fingers on the right hand are pressed), we press our Tenth Finger (left index finger) and clear all the fingers on the right hand at the same time. We call this manipulation, the *"Carry-over Manipulation"* or press 10, clear 9. To establish onety-one (11), we keep the Tenth Finger pressed, and go back to the units hand and press the First Finger. We continue in this manner until we have reached onety-nine (19). To establish twenty, clear all fingers on the right hand and at the same time press the Twentieth Finger while keeping the Tenth Finger pressed.

## 10

Press: 10 one-by-one using the manipulation of press 10, clear 9.*

Say: 1, 2, 3, 4, 5, 6, 7, 8, 9, onety.



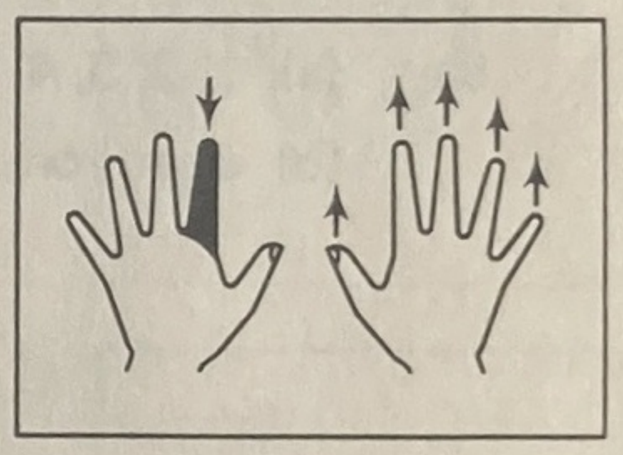
## 11

Press: 11 one-by-one using the manipulation of press 10, clear 9,* 11.

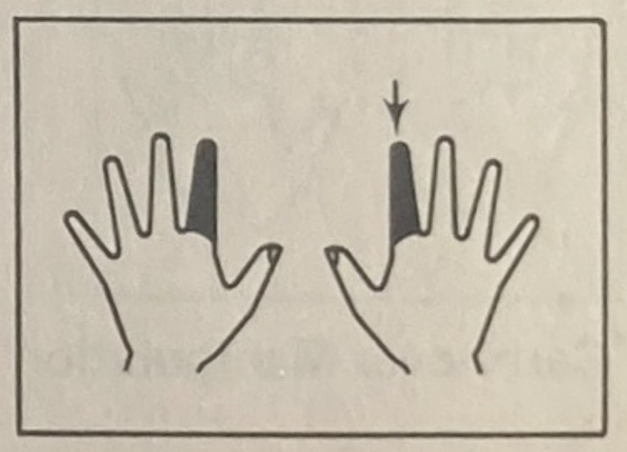
Say: (a) 1, 2, 3, 4, 5, 6, 7, 8, 9, onety.

(b) onety-one.

(a)



(b)



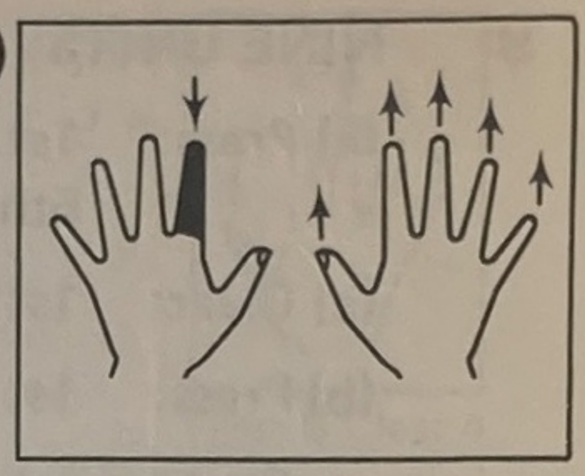
*Carry-over Manipulation
**Flip Manipulation
***The verbal instruction "by pressing 5 and clearing 4" can be eliminated once the Flip Manipulation is mastered.

**12** Press: 12 one-by-one using the manipulation of press 10, clear 9,* 11, 12.

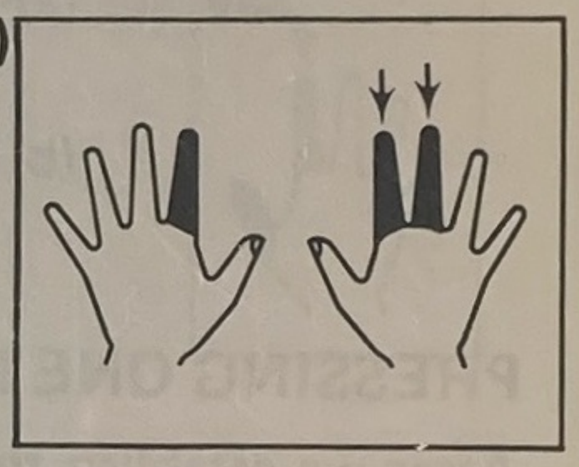
Say: (a) 1, 2, 3, 4, 5, 6, 7, 8, 9, onety.

(b) onety-one, onety-two.

(a)



(b)

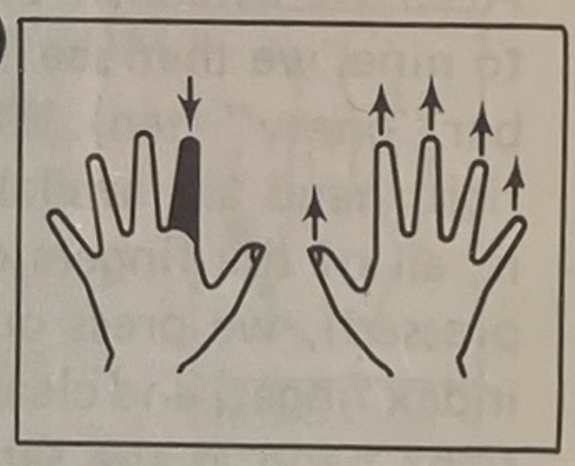


**13** Press: 13 one-by-one using the manipulation of press 10, clear 9,* 11, 12, 13.

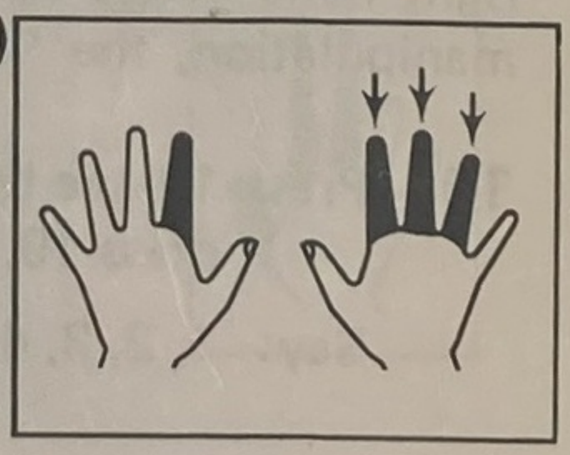
Say: (a) 1, 2, 3, 4, 5, 6, 7, 8, 9, onety.

(b) onety-one, onety-two, onety-three.

(a)



(b)

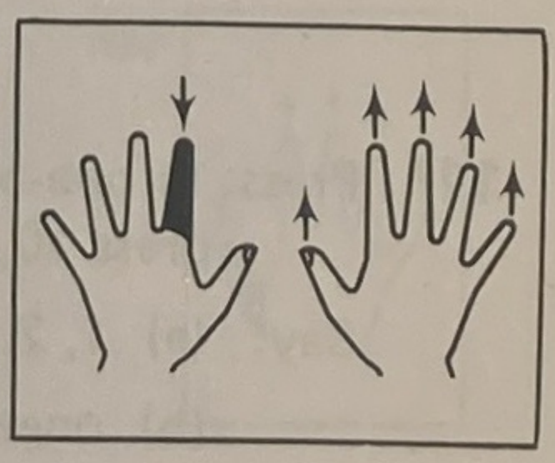


**14** Press: 14 one-by-one using the manipulation of press 10, clear 9,* 11, 12, 13, 14.

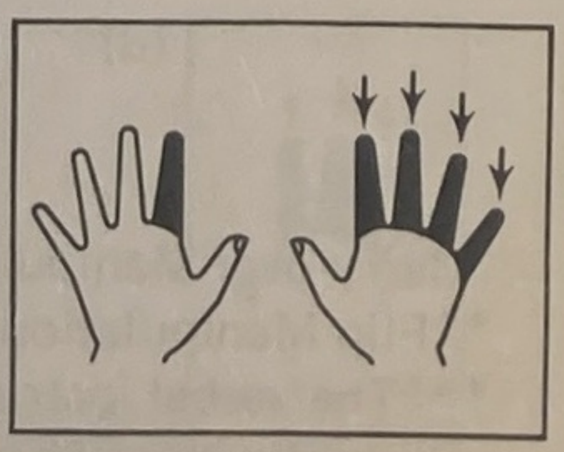
Say: (a) 1, 2, 3, 4, 5, 6, 7, 8, 9, onety.

(b) onety-one, onety-two, onety-three, onety-four.

(a)



(b)



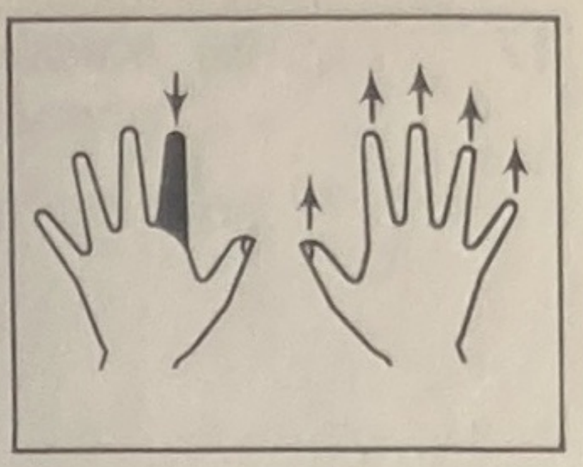
*Carry-over Manipulation

**15** Press: 15 one-by-one using the manipulation of press 10, clear 9,* 11, 12, 13, 14, (press 5 clear 4)** 15.

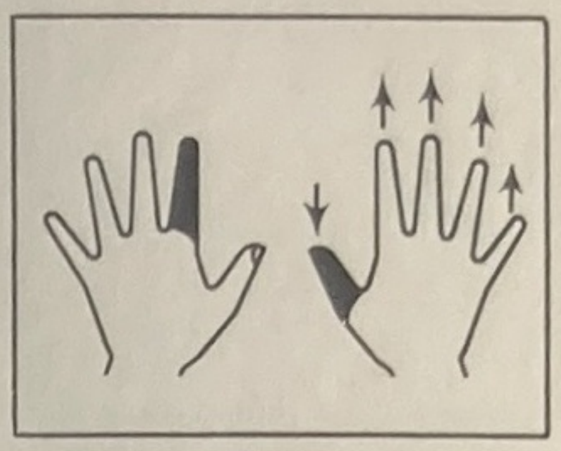
Say: (a) 1, 2, 3, 4, 5, 6, 7, 8, 9, onety.

(b) onety-one, onety-two, onety-three, onety-four, onety-five.

(a)



(b)



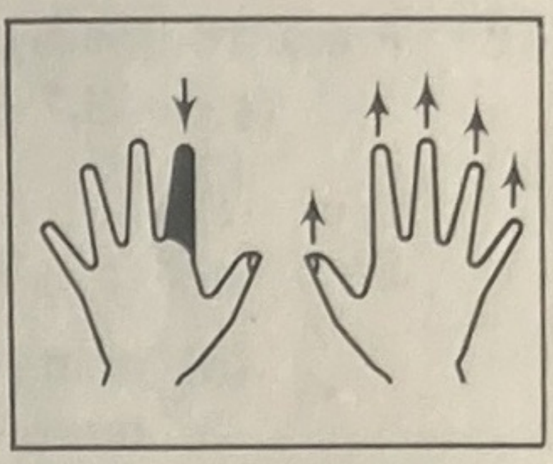
**16** Press: 16 one-by-one using the manipulation of press 10, clear 9,* 11, 12, 13, 14, (press 5 clear 4)** 15, 16.

Say: (a) 1, 2, 3, 4, 5, 6, 7, 8, 9, onety.

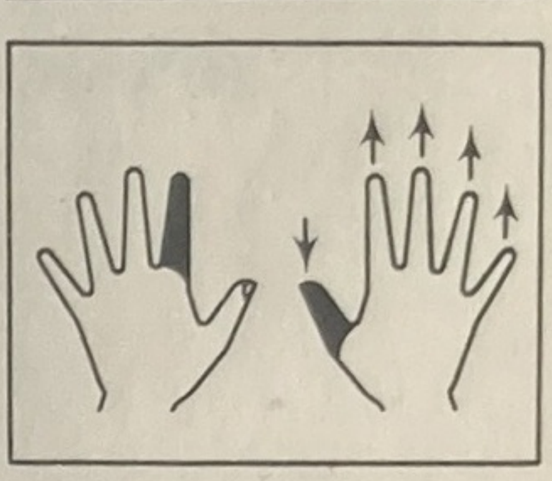
(b) onety-one, onety-two, onety-three, onety-four, onety-five.

(c) onety-six.

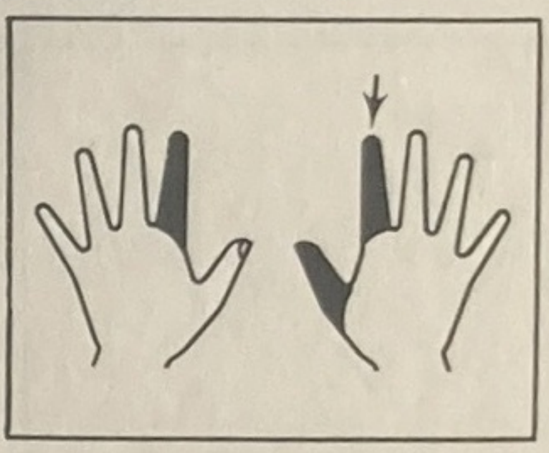
(a)



(b)



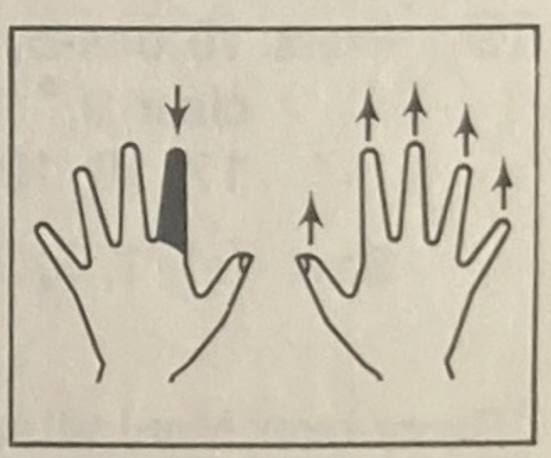
(c)



**17** Press: 17 one-by-one using the manipulation of press 10, clear 9,* 11, 12, 13, 14, (press 5 clear 4)** 15, 16, 17.

Say: (a) 1, 2, 3, 4, 5, 6, 7, 8, 9, onety.

(a)

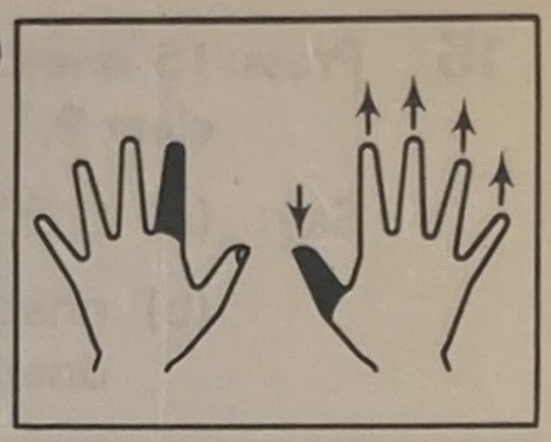


*Carry-over Manipulation
**Flip Manipulation

**17** (b) onety-one, onety-two, onety-three, onety-four, onety-five.

(c) onety-six, onety-seven.

(b)



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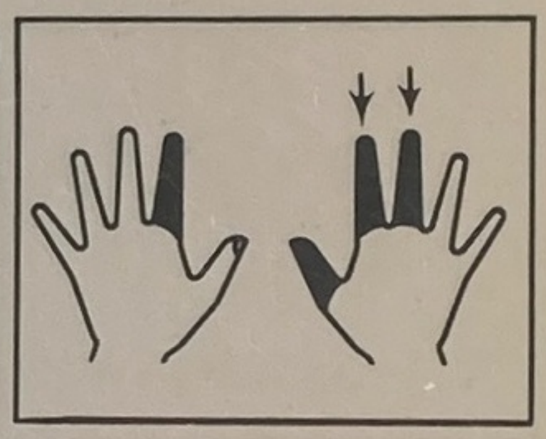


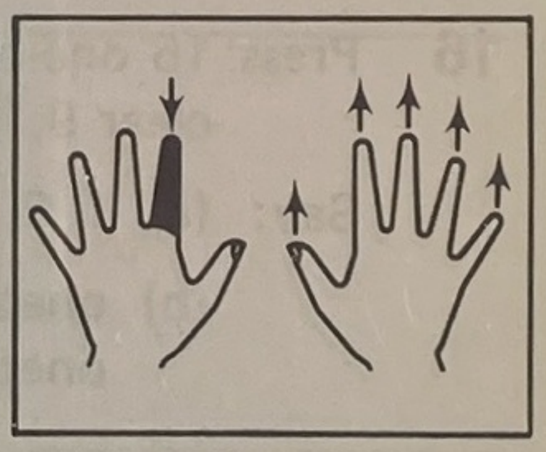
**18** Press: 18 one-by-one using the manipulation of press 10, clear 9,* 11, 12, 13, 14, (press 5 clear 4)** 15, 16, 17, 18.

Say: (a) 1, 2, 3, 4, 5, 6, 7, 8, 9, onety.

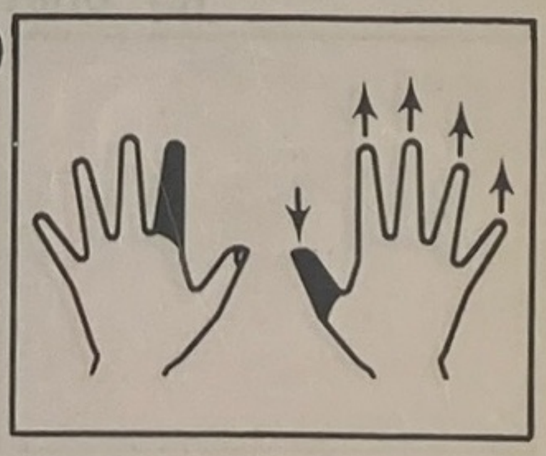
(b) onety-one, onety-two, onety-three, onety-four, onety-five.

(c) onety-six, onety-seven, onety-eight.

(a)



(b)



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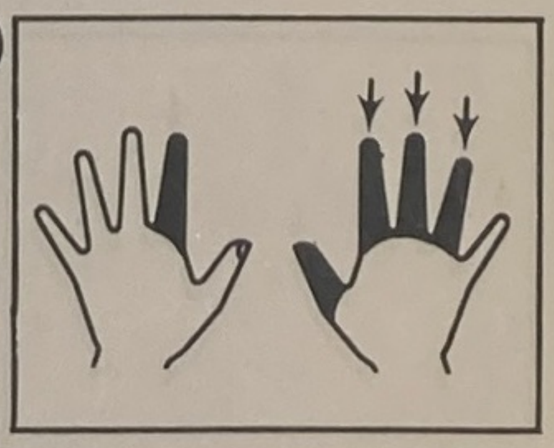
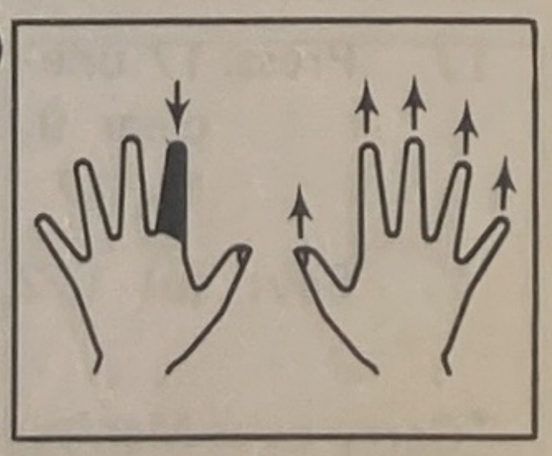


**19** Press: 19 one-by-one using the manipulation of press 10, clear 9,* 11, 12, 13, 14 (press 5 clear 4)** 15, 16, 17, 18, 19.

Say: (a) 1, 2, 3, 4, 5, 6, 7, 8, 9, onety.

(a)



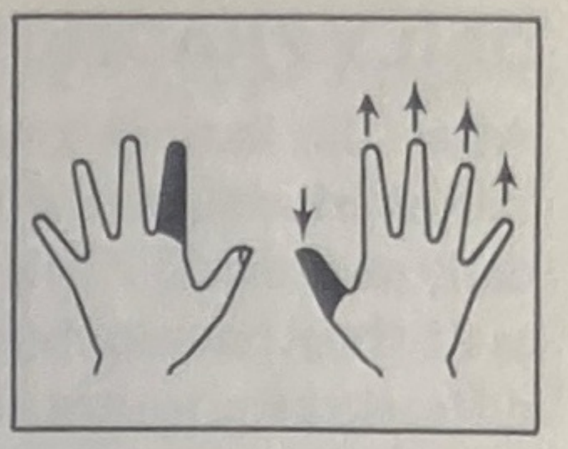
*Carry-over Manipulation
**Flip Manipulation

**19**

(b) onety-one, onety-two, onety-three, onety-four, onety-five.

(c) onety-six, onety-seven, onety-eight, onety-nine.

(b)



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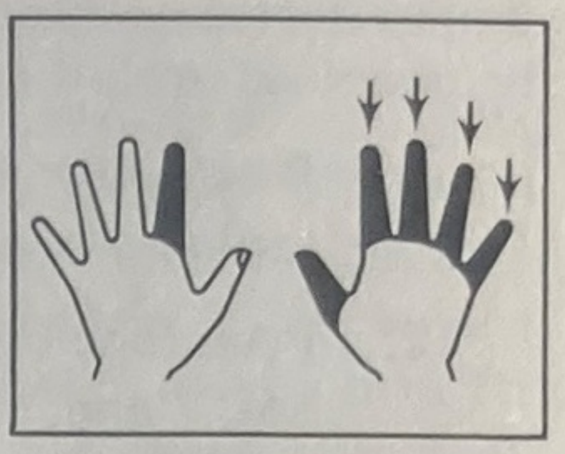


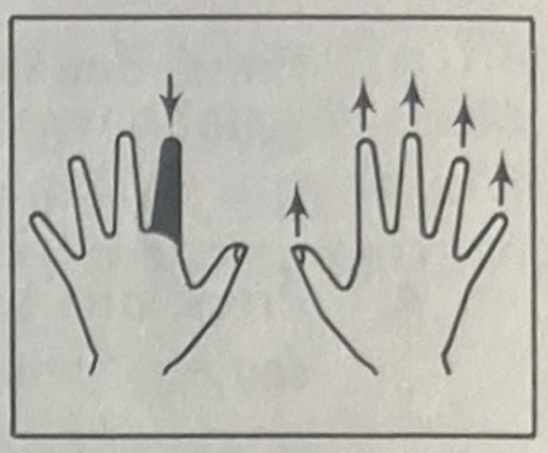
**20** Press: 20 one-by-one using the manipulation of press 10, clear 9,* 11, 12, 13, 14, (press 5 clear 4)** 15, 16, 17, 18, 19, (press 10 clear 9)* 20. This last manipulation requires you to press the 20th finger and clear 9 while the 10th finger remains pressed.

Say: (a) 1, 2, 3, 4, 5, 6, 7, 8, 9, onety.

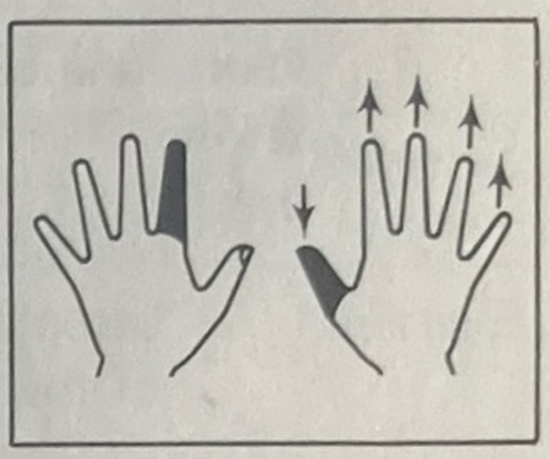
(b) onety-one, onety-two, onety-three, onety-four, onety-five.

(c) onety-six, onety-seven, onety-eight, onety-nine, twenty.

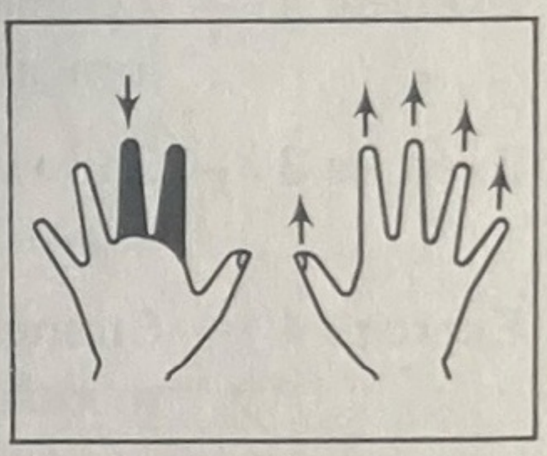
(a)



(b)



(c)



*Carry-over Manipulation
**Flip Manipulation

## DAILY PRACTICE EXERCISES

After the various numbers are firmly established, you can begin to practice certain daily exercises. While these exercises look as if they have only slight variations, the differences are very important and meant to instill certain key concepts. They should be practiced until they become routine.

1. Press, one by one, up to ten, saying; "one, two, three,...ten."
2. Press, one by one, up to ten, saying, "one, and one more, and one more...and one more."
3. Press, one by one, up to twenty, saying; "one, two, three,...ten, one, two, three,...ten."
4. Press, one by one, up to twenty, saying, "one, two, three,...onety, onety-one, onety-two,...twenty."
5. Press, one by one, up to twenty, saying; "one, two, one, two,... one, two."

Exercise 1 — Strengthens the ordinality of the numbers.

Exercise 2 — Teaches the concept of the integers system.

Exercise 3 — Shows how the left hand grows by ten.

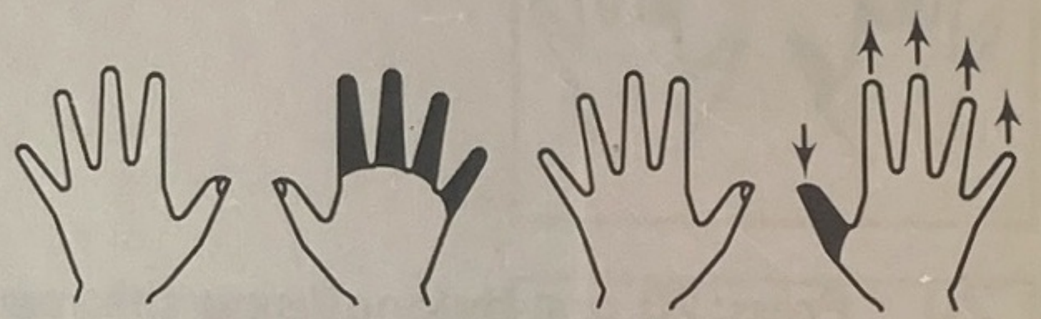
Exercise 4 — Strengthens the onety nomenclature.

Exercise 5 — Prepares the student for multiplication.

## TWO KEY MANIPULATIONS

There are only 2 key manipulations to learn in stage 1. Once they are learned, even young children can perform problems that will amaze adults. It is therefore useful to point out what these manipulations are and to suggest frequent daily practice. The manipulations are 4 + 1 (Press 5, Clear 4) and 9 + 1 (Press 10, Clear 9). When you add 1 to 4, your fingers must press 5 and clear 4 simultaneously.

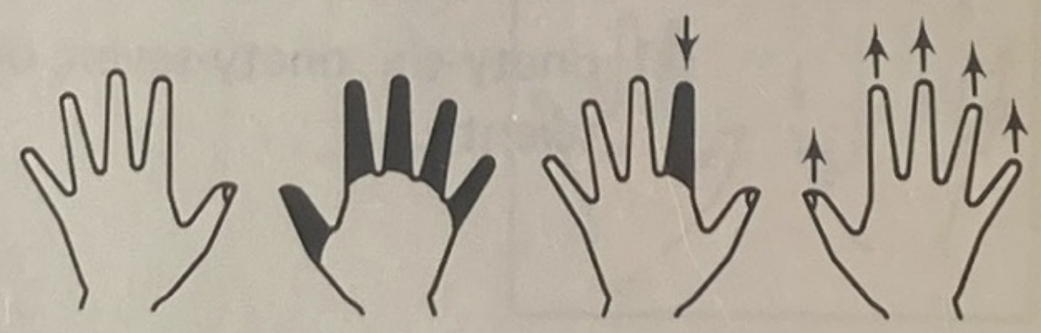
**4 + 1 = 5**



**(Press 5, Clear 4)**

When you add 1 to 9, your fingers must press 10 and clear 9 simultaneously.

**9 + 1 = 10**



**(Press 10, Clear 9)**

With each of these manipulations a single finger must be pressed and at the same time others cleared. Study these manipulations carefully and practice them daily.

These manipulations are crucial. They will be learned more quickly if teachers inflect their voices when they come to these maneuvers. Thus when adding 1 to 4 or 1 to 9, the voice should be raised as the manipulation is made. Moreover, with very young children it is extremely helpful to exaggerate the manipulations.

Thus when pressing 5 and clearing 4, wiggle all the cleared fingers in the air, and when pressing 10 and clearing 9, wave the entire hand in the air. Once these manipulations are understood, the exaggerated motions and tone inflections can be diminished.

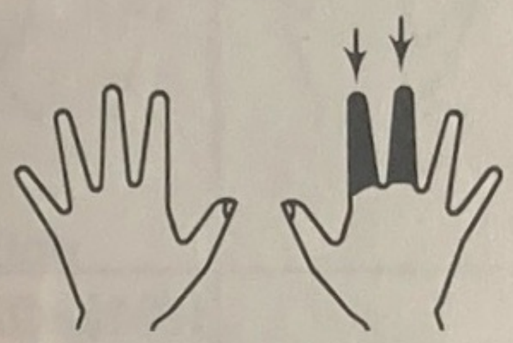
## ADDING 2 NUMBERS

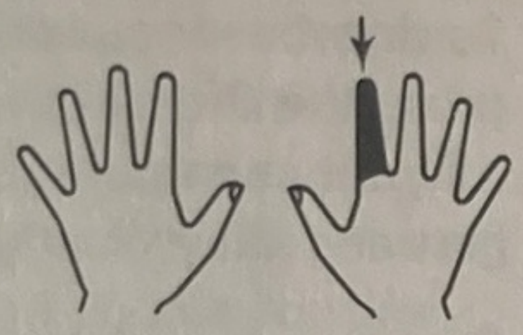
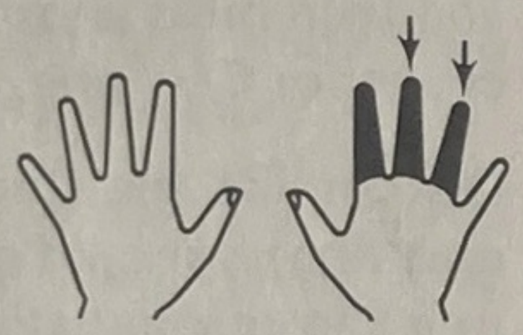
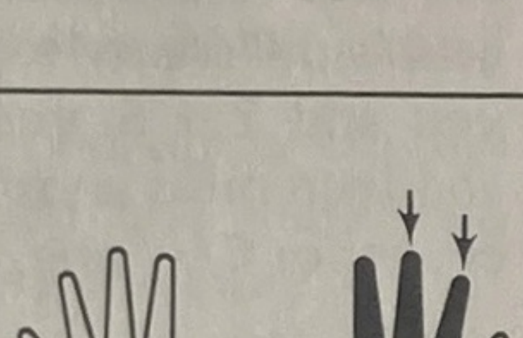
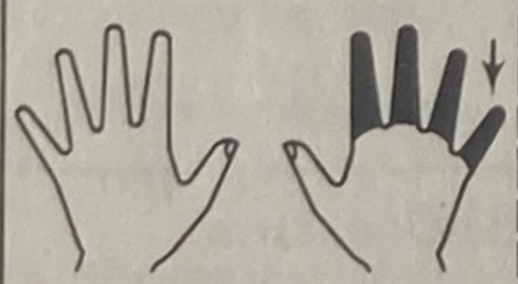
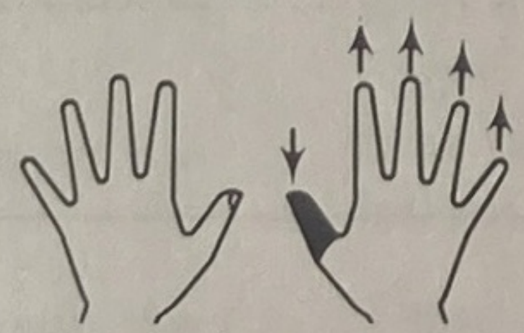
Adding numbers in Chisanbop ™ is almost simplicity itself. All one does is press the first number, one by one, and then while keeping the first number pressed, press the second number, one by one. Once the second number is pressed, the sum is known by reading the fingers i.e., verbalizing the value registered on the fingers. Thus in adding 2 + 2, you first press 2, one by one, leaving you with the 1st and 2nd fingers pressed. You then press another 2, one by one, and to do this you must press the 3rd and 4th fingers. At the end of your manipulations you read your fingers and they clearly show the answer is 4.

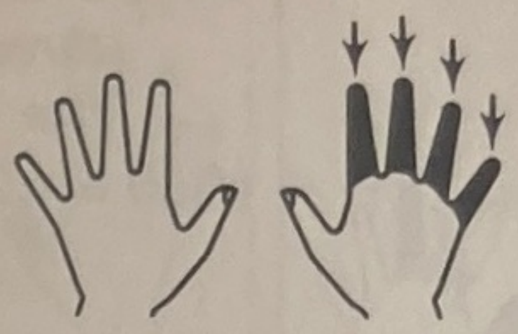
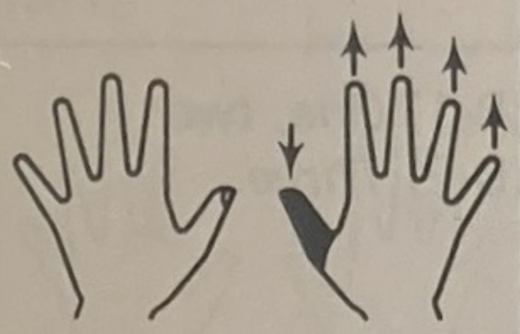
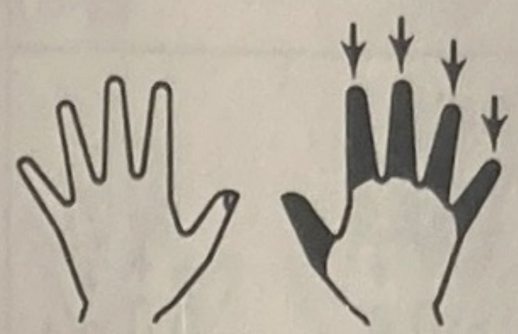
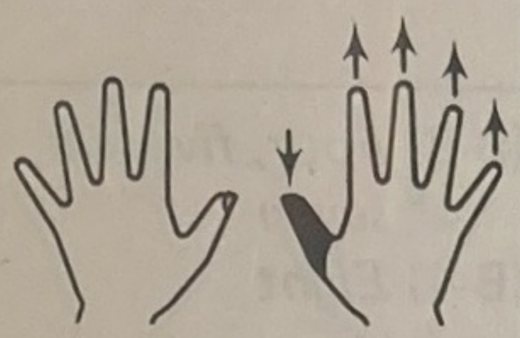
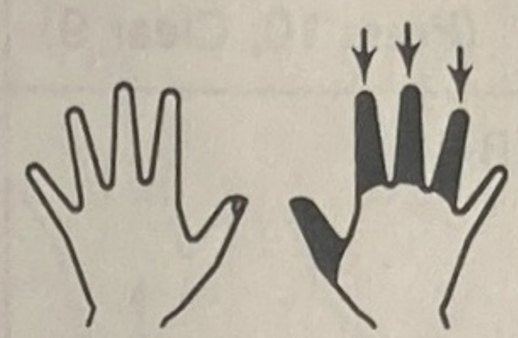
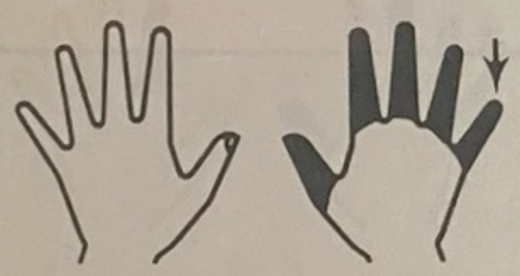
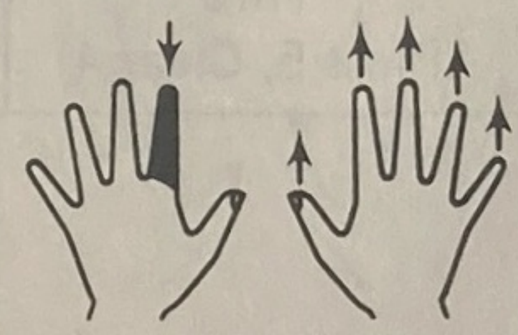
Caution should be used when you press the second addend. The value of the fingers is different from the verbal count. If you add 2 + 5, you press 2, one-by-one, and then press another 5, one-by-one. The count for 5 is "one, two, three, four, five."

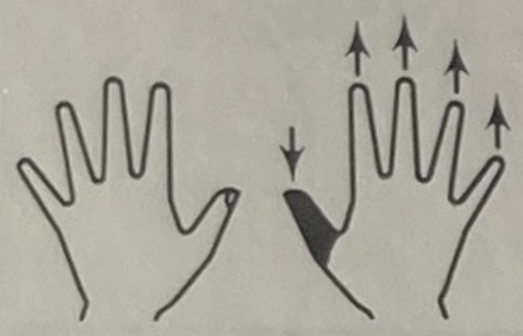
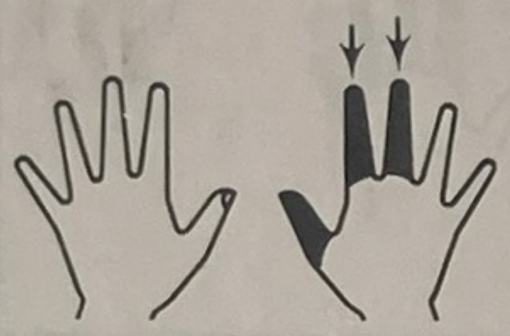
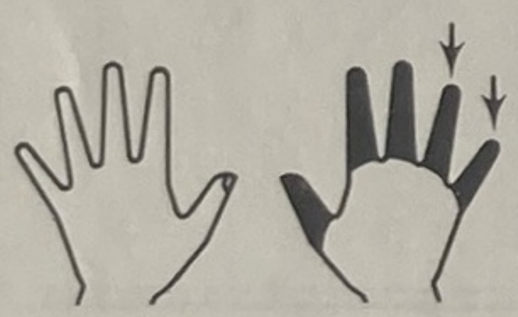
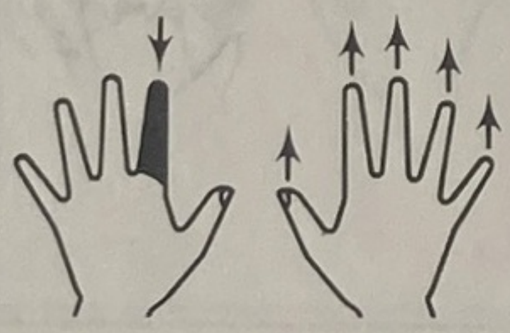
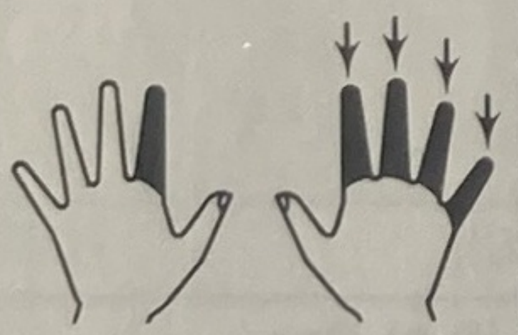
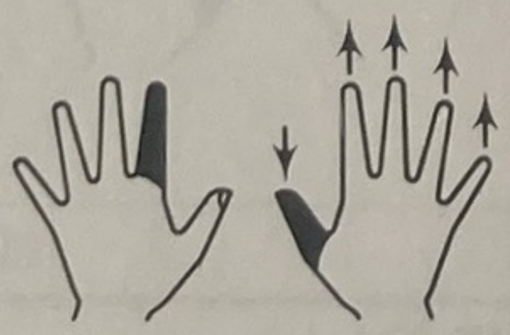
Do not lift the fingers prematurely. The unit fingers should never be cleared except in a flip manipulation. The fifth finger is only cleared in a carry-over manipulation.

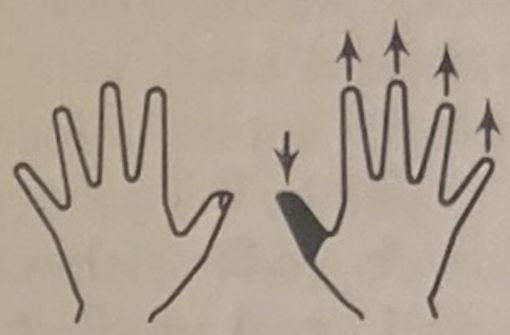
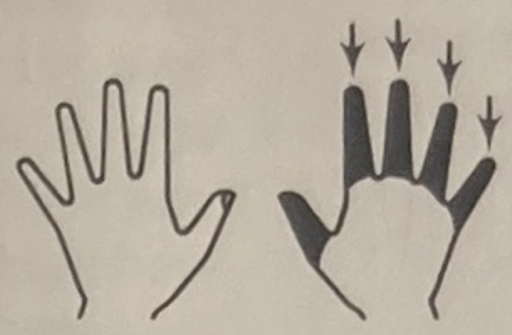
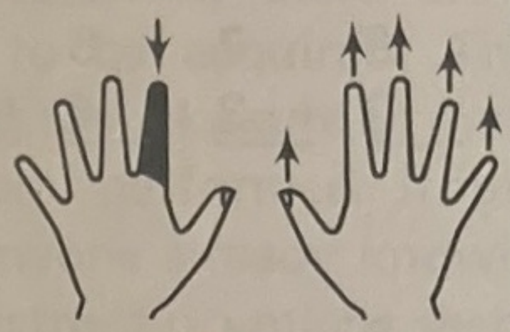
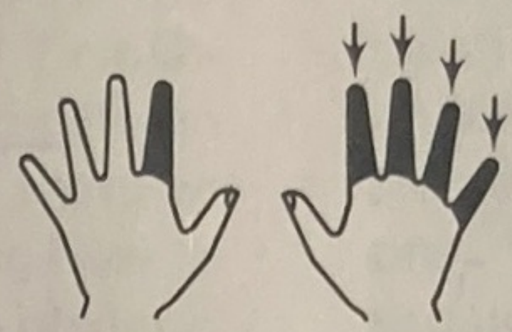
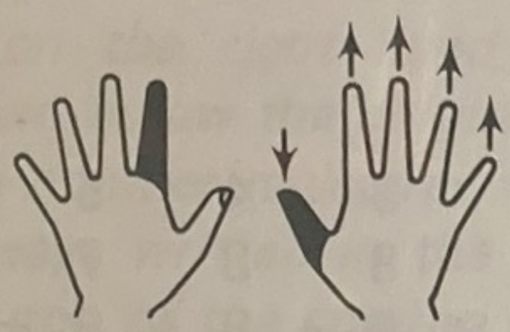
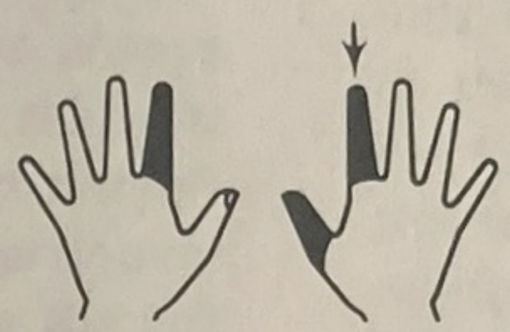
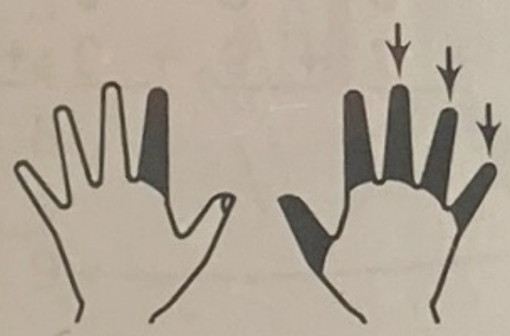
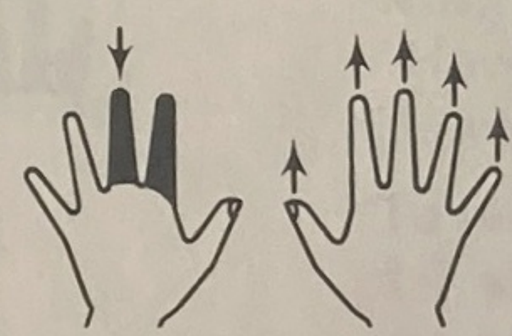
## ADDITION STAGE 1

| Problem | Finger Manipulations (Press One By One) | Verbal Count | Fingers Read |
|---|---|---|---|
| (A) 2 | (A)  | (A) *One, two.* | |
| (B) + 2 | (B)  | (B) *One, two.* | (4) |

| Problem | Finger Manipulations (Press One By One) | Verbal Count | Fingers Read |
|---|---|---|---|
| (A)<br>1<br>(B)<br>+ 2 | (A)<br>  <br>(B)<br>  | (A) *One*<br>(B) *One, two* | (3) |
| (A)<br>3<br>(B)<br>+ 2 | (A)<br>  <br>(B-1)<br>  <br>(B-2)<br>  <br>Through<br>(Press 5, Clear 4) | (A) *One, two, three*<br>(B-1) *One*<br>(B-2) *Two* | (5) |

| Problem | Finger Manipulations (Press One By One) | Verbal Count | Fingers Read |
|---|---|---|---|
| (A)<br><br>4 | (A)<br>  | (A) *One, two, three, four* | |
| (B)<br><br>+ 5 | (B-1)<br>  <br>Through<br>(Press 5, Clear 4)<br>(B-2)<br>  | (B-1) *One*<br>(B-2) *Two, three, four, five* | (9) |
| (A)<br><br>8 | (A-1)<br>  <br>Through<br>(Press 5, Clear 4)<br>(A-2)<br>  | (A-1) *One, two, three, four, five*<br>(A-2) *Six, seven, eight* | |
| (B)<br><br>+ 2 | (B-1)<br>  <br>(B-2)<br>  <br>Through<br>(Press 10, Clear 9) | (B-1) *One*<br>(B-2) *Two* | (10) |

| Problem | Finger Manipulations (Press One By One) | | Verbal Count | Fingers Read |
|---|---|---|---|---|
| (A)<br>7 | (A-1)<br>  <br>Thru<br>(Press 5, Clear 4) | (A-2)<br>  | (A-1) *One, two, three, four, five*<br>(A-2) *Six, seven* | |
| (B)<br>+ 8 | (B-1)<br>  | (B-2)<br>  <br>Thru<br>(Press 10, Clear 9) | (B-1) *One, two*<br>(B-2) *Three* | |
| | (B-3)<br>  | (B-4)<br>  <br>Thru<br>(Press 5, Clear 4) | (B-3) *Four, five, six, seven*<br>(B-4) *Eight* | (15) |

| Problem | Finger Manipulations (Press One By One) | Verbal Count | Fingers Read |
|---|---|---|---|
| (A)<br>9 | (A-1)  Thru (Press 5, Clear 4)<br>(A-2)  | (A-1) *One, two, three, four, five*<br>(A-2) *Six, seven, eight, nine* | |
| (B)<br>7 | (B-1)  Thru (Press 10, Clear 9)<br>(B-2)  | (B-1) *One*<br>(B-2) *Two, three, four, five* | |
| | (B-3)  Thru (Press 5, Clear 4)<br>(B-4)  | (B-3) *Six*<br>(B-4) *Seven* | |
| (C)<br>+ 4 | (C-1)  <br>(C-2)  Thru (Press 10, Clear 9) | (C-1) *One, two, three*<br>(C-2) *Four* | (20) |

## EXERCISES

A.
$$\begin{array}{rrrrr} 1 & 1 & 1 & 2 & 3 \\ +1 & +2 & +3 & +2 & +1 \end{array}$$

$$\begin{array}{rrrrr} 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 0 & 3 & 1 \\ +1 & +0 & +1 & +0 & +2 \end{array}$$

B.
$$\begin{array}{rrrrr} 5 & 4 & 1 & 3 & 2 \\ +0 & +1 & +4 & +2 & +3 \end{array}$$

$$\begin{array}{rrrrr} 4 & 3 & 1 & 2 & 2 \\ 0 & 1 & 3 & 1 & 0 \\ +1 & +1 & +1 & +2 & +3 \end{array}$$

C.
$$\begin{array}{rrrrr} 5 & 1 & 4 & 3 & 2 \\ +1 & +6 & +4 & +6 & +7 \end{array}$$

$$\begin{array}{rrrrr} 3 & 1 & 1 & 4 & 4 \\ 2 & 3 & 7 & 1 & 2 \\ +2 & +2 & +1 & +1 & +3 \end{array}$$

D.
$$\begin{array}{rrrrr} 9 & 8 & 7 & 6 & 5 \\ +1 & +2 & +3 & +4 & +5 \end{array}$$

$$\begin{array}{rrrrr} 6 & 5 & 4 & 3 & 1 \\ 1 & 2 & 2 & 4 & 5 \\ +3 & +3 & +4 & +3 & +4 \end{array}$$

E.
$$\begin{array}{rrrrr} 9 & 9 & 9 & 8 & 8 \\ +9 & +8 & +7 & +8 & +9 \end{array}$$

$$\begin{array}{rrrrr} 9 & 4 & 8 & 4 & 3 \\ 1 & 5 & 9 & 9 & 8 \\ +9 & +7 & +1 & +4 & +7 \end{array}$$

F.
$$\begin{array}{rrrrr} 2 & 3 & 4 & 5 & 6 \\ 9 & 8 & 8 & 7 & 7 \\ +9 & +9 & +8 & +8 & +7 \end{array}$$

$$\begin{array}{rrrrr} 7 & 9 & 7 & 9 & 8 \\ 7 & 6 & 8 & 9 & 9 \\ +8 & +7 & +9 & +9 & +8 \end{array}$$

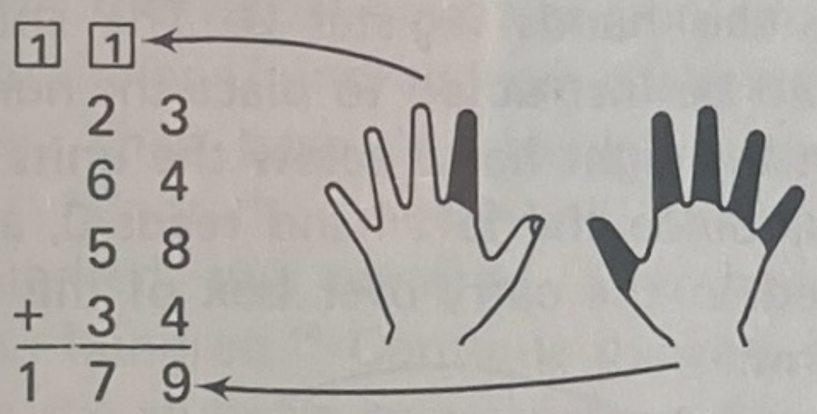
## MULTI-DIGIT ADDITION

One of the outstanding features of Chisanbop (TM) is that even in Stage 1, young children can add more than one column. In fact, once they have mastered the two difficult manipulations discussed in the last section, they are willing to attack any series of numbers. They do not care how many rows or how many columns are involved in the problem. They plunge right in to the calculations with great confidence ignoring the degree of difficulty.

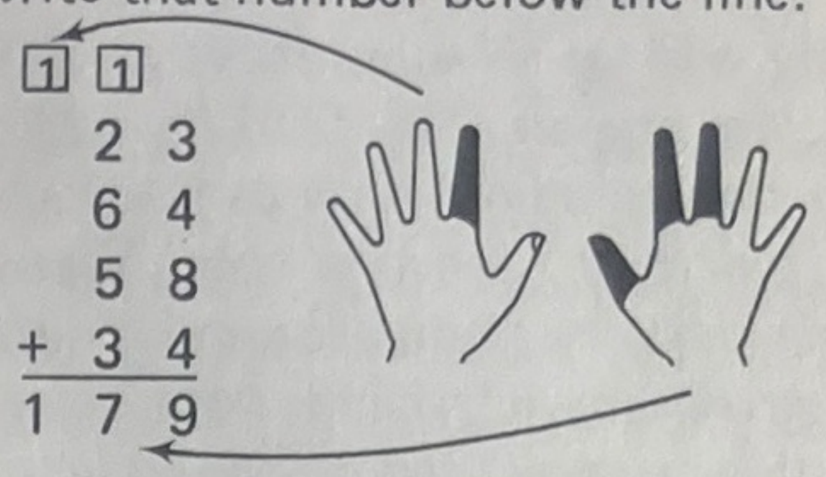
In adding columns, there are only two new skills to be acquired. The first requirement is to make sure all multi-column problems are performed from right to left. To someone already knowing how to add, this instruction might seem obvious, but if you are instructing a youngster just beginning to add, it is not at all obvious.

The second skill involved is recognizing where to place the answers. *The number registered on the right hand is always written down below the column that was added. The number registered on the left hand is always written in the carry-over box at the top of the column to the left of the column that was added.*

EXAMPLE:




After adding the units' column, unit by unit, the hands register 19. A right-handed child should be instructed to first read his right hand and then write that number below the units' column. He should then read his left hand and place the number (one 10) into the carry-over box in the tens' column. A left-handed child should reverse the order of writing and write the number from his left hand into the carry-over box first, and then read his right hand and write that number below the line.




After completing the units' column, the tens' column is added. At this point, an important lesson in place value can be learned. Instead of calling the numbers one, two, six, five and three, the child should be told he is counting tens on his right hand and call the numbers in the tens' column one ten, two tens, six tens, five tens, and three tens. As in all prior examples these numbers are added unit by unit, one finger at a time. The oral count, as these numbers are added, should be by tens. Thus when pressing a six from the tens' column on the right hand, the number should be counted as "onety, twenty, thirty, forty, fifty, sixty." At the end of the calculation the hands should register 17 i.e. onety-seven tens. Since this is the first time the right hand is counting tens, teachers should ask their students: "How many tens on the right hand? How many hundreds on the left hand?"

To reinforce the use of the carry-over box a 7 should be written below the line in the tens' column and a 1 placed in the carry-over box of the hundreds' column.

Since there are no other numbers in the hundreds' column, the 1 is brought down below the line. When the notion of carry-over is well established, the use of a box

in a column with no other number can be dropped and the answer from both left and right hands written below the line directly.

The multi-digit examples in this book begin with columns which do not individually add up to sums above nine. However, to prepare the child to add more than one column with a carry-over number, we must have him enter a zero 0 in the carry-over box because no numbers appear on the left hand. After examples like this are practiced, the child will compute examples with totals beyond 9. At this point he will have to write down the number that is registered on his left hand in the carry-over box provided. After this technique is mastered, the child learns a new skill which no longer requires use of the carry-over box. In Stage 2, the carry-over numbers are transferred from the left hand to the right hand instead of using the carry-over box.

## EXERCISES

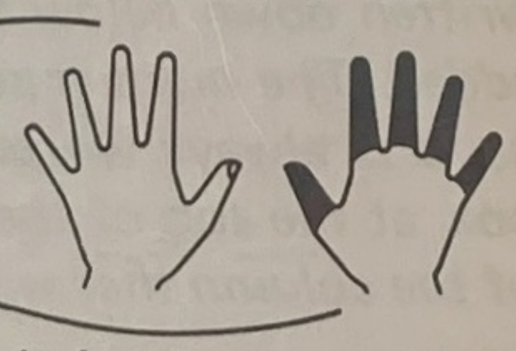
| | | | |
|---|---|---|---|
| □□ | □□ | □□ | □□ |
| 63 | 72 | 26 | 17 |
| 14 | 12 | 30 | 41 |
| +21 | +15 | +42 | +31 |
| □□ | □□ | □□ | □□ |
| 41 | 3 | 5 | 82 |
| 23 | 62 | 40 | 3 |
| +4 | +21 | +33 | +12 |

1. For the Right-Handed Child—After the child has added the ones' column, he should write down the number registered on his Right Hand below the line in the ones' column. Since, in these examples, there is no number registered on the Left Hand, he writes zero in the carry-over box. He continues with this procedure working from right to left columns.
2. For the Left Handed Child—The Left-Handed child follows a similar procedure with one important exception. He always writes the number to go in the carry-over box (i.e. the number registered on the Left Hand) first, while keeping the Right Hand Pressed. He then writes the number registered on the Right Hand below the line on the ones' column. He continues with this procedure working from right to left columns.
3. The numbers registered on the Left Hand belong to the carry-over box while the numbers on the Right Hand belong to the total sum for each column.

### Changing the Value of the Hands

(A) □□□0

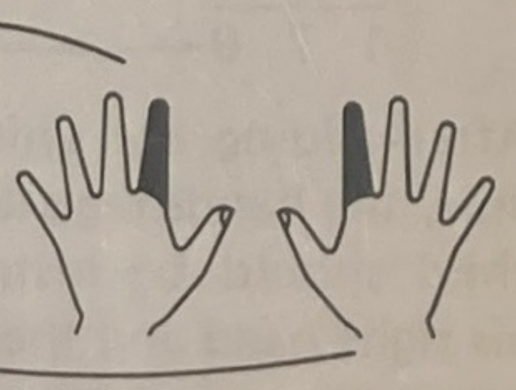
9,483
+ 5,236
9



After adding the units' column one-by-one, the hands register 9. The student should be instructed to place the number from the right hand below the units' column. Since the left hand reads 0, a 0 is placed in the carry-over box of the tens' column.

(B) □□10

9,483
+ 5,236
19

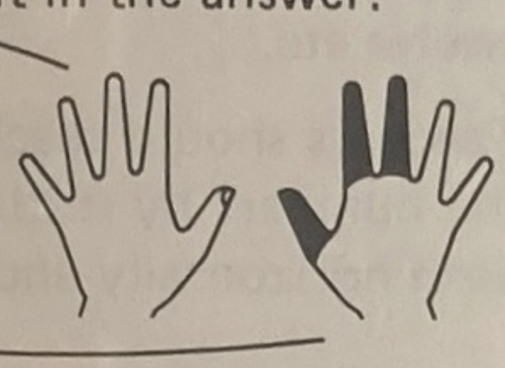


As you move to the next column both hands take on different values. You must

make this explicit by the way you count the numbers aloud as you press the fingers, one-by-one. Thus in calculating the tens' column, the right hand is counting tens and the left hand is counting hundreds. As the students continue, they press the 8, one-by-one, but you count it as: "onety, twenty,...,eighty." So too, when they press the three, you must verbalize it as: "onety, twenty, thirty." When you have completed the calculation, you must ask the students: "How many tens on the right hand?" The answer, "One," is written below the line in the tens' column. You then must ask them: "How many hundreds on the left hand?" The answer: "One," is written into the carry-over box of the hundreds' column. If you continue to verbalize in this manner, they will have no trouble in knowing where to place the numbers. They also will begin to see how the left hand is always a carry-over number and the right hand represents the proper digit in the answer.

(C)

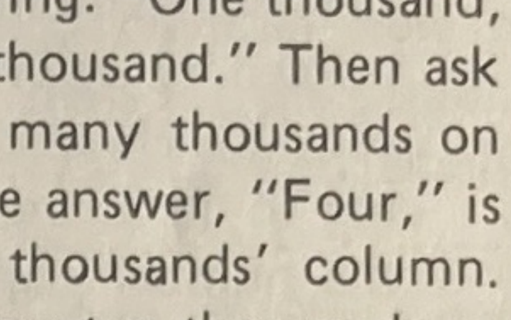
| | | | | |
|---|---|---|---|---|
| ☐ | 0 | 1 | 0 | |
| | 9, | 4 | 8 | 3 |
| + | 5, | 2 | 3 | 6 |
| | | 7 | 1 | 9 |



Continue with the hundreds' column. Emphasize the verbal count as you proceed with the one hundred from the carry-over box. Then have the students press the four hundreds, one-by-one, saying: "One hundred, two hundred, three hundred, four hundred." Continue the verbal count as the students press the two hundreds, one-by-one. When the calculation is complete, ask the students: "How many hundreds on the right hand?" The answer "Seven" is written below the hundreds' column. Then ask: "How many thousands on the left hand?" The answer, "Zero," is written in the carry-over box above the thousands' column.

(D)

| | | | | |
|---|---|---|---|---|
| 1 | 0 | 1 | 0 | |
| | 9, | 4 | 8 | 3 |
| + | 5, | 2 | 3 | 6 |
| | 4, | 7 | 1 | 9 |



As you proceed to the next column, re-emphasize the verbal count. Have the students press the nine thousands, one-by-one saying: "One thousand, two thousand...nine thousand." Continue the press with the students adding the five thousands, one-by-one saying: "One thousand, two thousand...five thousand." Then ask the students: "How many thousands on the right hand?" The answer, "Four," is written below the thousands' column. Then ask: "How many ten thousands on the left hand?" The answer, "One," is written in the carry-over box above the ten thousands' column.

(E)

| | | | | |
|---|---|---|---|---|
| 1 | 0 | 1 | 0 | |
| | 9, | 4 | 8 | 3 |
| + | 5, | 2 | 3 | 6 |
| 1 | 4, | 7 | 1 | 9 |

Since there are no other numbers in the hundred thousands' column, the 1 is brought down below the line.

## EXERCISES

| | | |
|---|---|---|
| ☐☐☐ | ☐☐☐ | ☐☐☐ |
| 956 | 510 | 934 |
| 437 | 216 | 687 |
| +154 | +895 | +311 |
| ☐☐☐ | ☐☐☐ | ☐☐☐ |
| 789 | 608 | 129 |
| 698 | 219 | 950 |
| +776 | +434 | +621 |

1. For the Right-Handed Child—After the child has added the ones' column, he should first write down the number registered on his Right Hand below the line in the ones' column while keeping the Left Hand pressed. The child then reads the Left Hand and places the number in the carry-over box above the tens' column. He continues with this procedure working from right to left columns.
2. For the Left-Handed Child—The Left-Handed child follows a similar procedure with one important exception. He always writes the number to go in the carry-over box (i.e. the number registered on the Left Hand) first, while keeping the Right Hand Pressed. He then writes the number registered on his Right Hand below the line in the ones' column. He continues with this procedure working from right to left columns.
3. The numbers registered on the Left Hand belong to the carry-over box while the numbers on the Right Hand belong to the total sum for each column.

## Chisanbop ™ and Place Value

By now, you are beginning to see how Chisanbop ™ and place value complement one another so neatly. Whenever you move from the right hand to the left hand, you know that you are going to another group. If you have 9 units on the right hand and then you add one more, you must reach a 10. Chisanbop ™ has also created several teaching aids to emphasize this concept. If you have slower students in your class who are not gaining insight into place value, you may want to create the following visual aids for those students to use. The visual aids are grouped together at the end of the following explanations.

## DECIMAL NUMBER CHART

The first visual aid is the number chart. You will notice that in each line the last number ends in 9. Some people might be tempted to think that the number ten or twenty is the final number of a group, but they really are the first numbers of new groups. It is in moving from 9 to 10 that one moves from units to tens.

On the number chart we have incorporated the Chisanbop ™ terminology for the numbers 10-19. We can not overemphasize the importance of these new names for very young children when they are first learning their numbers. Later on, once the numbers are established, they return to the usual terms of ten, eleven, twelve etc.

Teachers should teach the relationship of the numbers by reading the number chart both horizontally and vertically.

## HYPAI Decimal Number Stairs.

Chisanbop ™ suggests an alternative to the number line. The first illustration is that of the Decimal Number Stairs. In making the number line rise vertically whenever another group of ten is reached, the children can see dramatically, the nature of the decimal system. They can begin to understand that the highest unit attainable is 9 and once ten is reached, they have moved to a new level. While discussing the number stairs, you can relate the change in levels to the carry-over

manipulation (press 10 clear 9). The physical and exaggerated finger manipulations of Chisanbop ™ along with the visual number stairs will help to instill the nature of the decimal system.

In a text book it is impossible to picture these number stairs without breaks. However, it is suggested that in the classroom you display a set of stairs from 0 to 99 which would run on without discontinuations.

The following illustrations depict the number stairs with discontinuations after 9, 19, 29, etc. The purpose of this is to show the relationships between the numbers. Thus you can compare for instance 73 and 53. 73 is 7 groups of 10's and 3 units while 53 is 5 groups of 10's and 3 units.

## HYPAI Chisanbop ™ Number Stairs

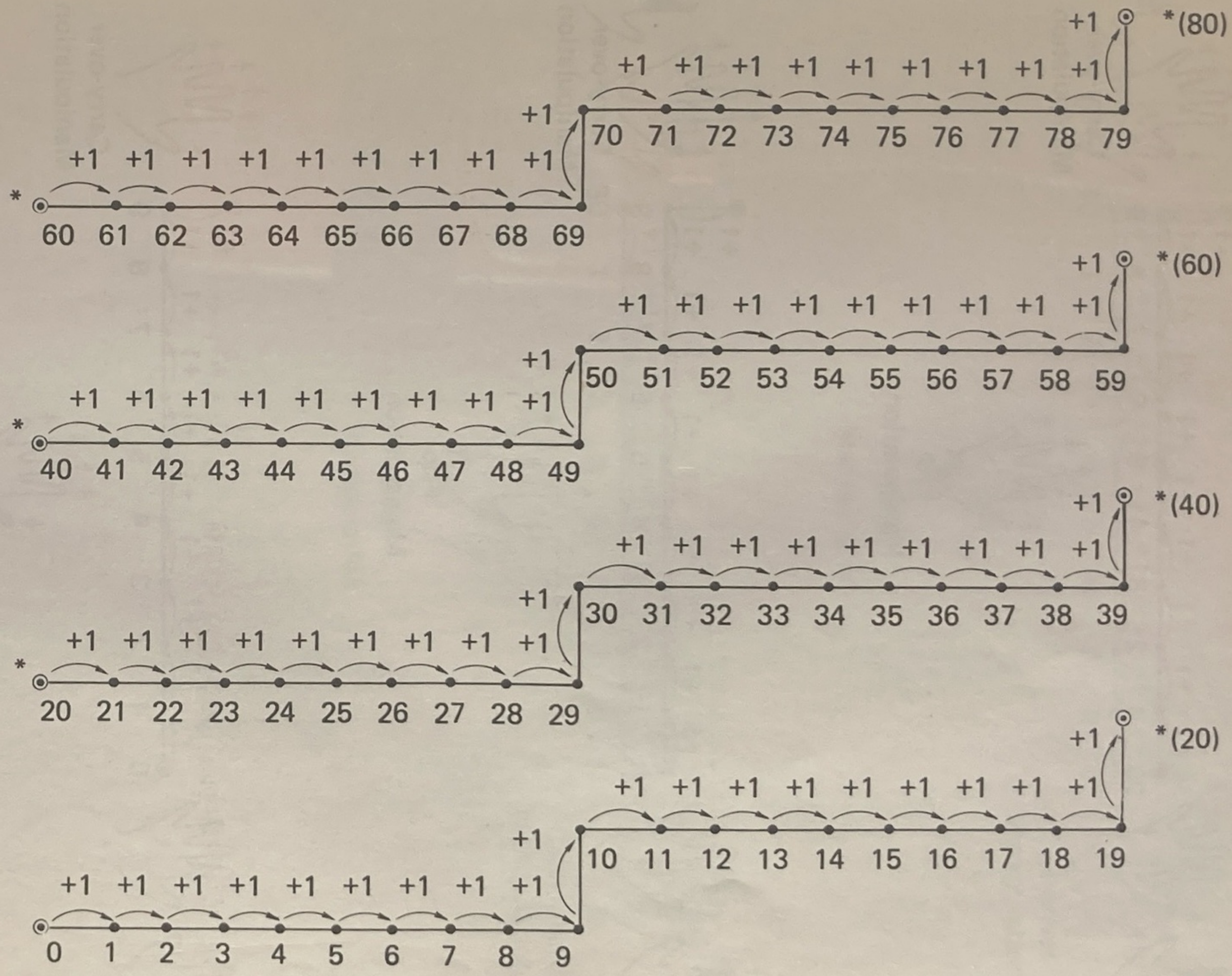
A second set of illustrations have been provided which we call the Chisanbop ™ Number Stairs. The reason they are Chisanbop ™ Number Stairs and not decimal stairs is because they also rise whenever moving from 4 to 5, 14 to 15 etc. Thus they provide an illustration that can complement the Flip Manipulation of press 5, clear 4. One more point can be made. When the number 7 is pressed, it is read at least subconsciously as 5, 6, 7. That is to say that we first count the thumb and then the index and middle fingers. Also in counting coins, e.g. a nickel and 2 pennies, we would count them as 5, 6, 7. Thus you can use the Chisanbop ™ stairs to show young children how to count. If 10 is the basis for the decimal system, it is safe to say that pragmatically, 5 is the sub-basis.

The Chisanbop ™ Number Stairs continue to rise when moving from 9 to 10, 19 to 20, etc. They fully include all the theory of the decimal stairs but add to the decimal system, the importance of the number 5.

## DECIMAL NUMBER CHART

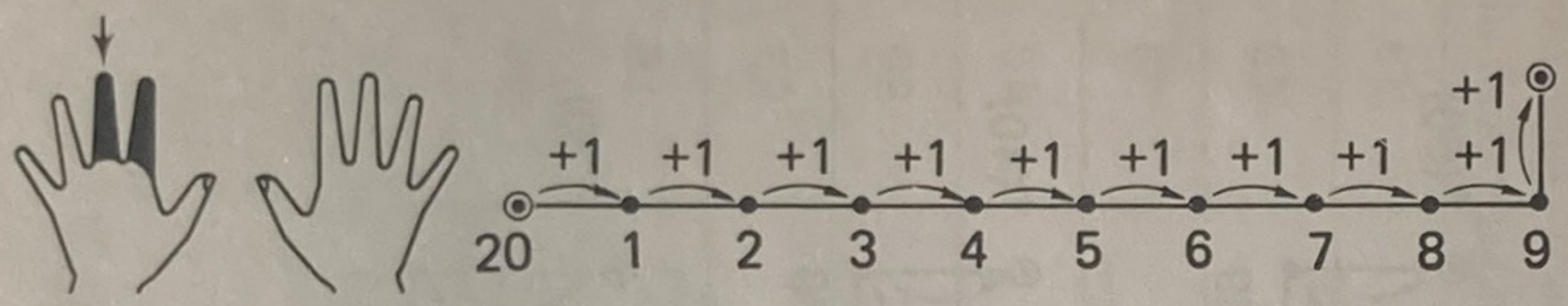
| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 |
| 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 |
| 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 |
| 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 |
| 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 |
| 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 |
| 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 |
| 20 | 21 | 22 | 23 | 24 | 45 | 46 | 47 | 48 | 49 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| onety (ten) | onety-one (eleven) | onety-two (twelve) | onety-three (thirteen) | onety-four (fourteen) | onety-five (fifteen) | onety-six (sixteen) | onety-seven (seventeen) | onety-eight (eighteen) | onety-nine (nineteen) |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

# HYPAI DECIMAL NUMBER STAIRS



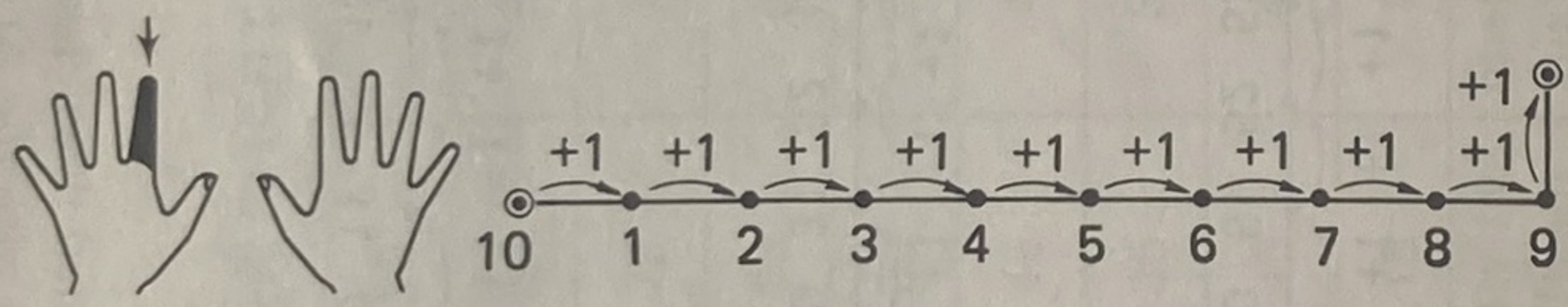

* Space restrictions force the use of 4 discontinued lines. Ideally, in the classroom, these stairs would be displayed as one continuous line with the 20's, 40's and 60's joined.

# HYPAI DECIMAL NUMBER STAIRS



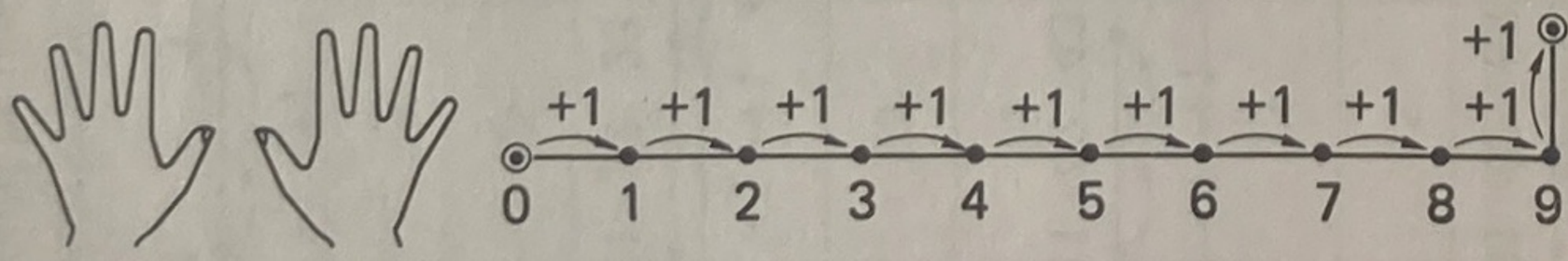

Carry-over Manipulation

Flip Manipulation




Carry-over Manipulation

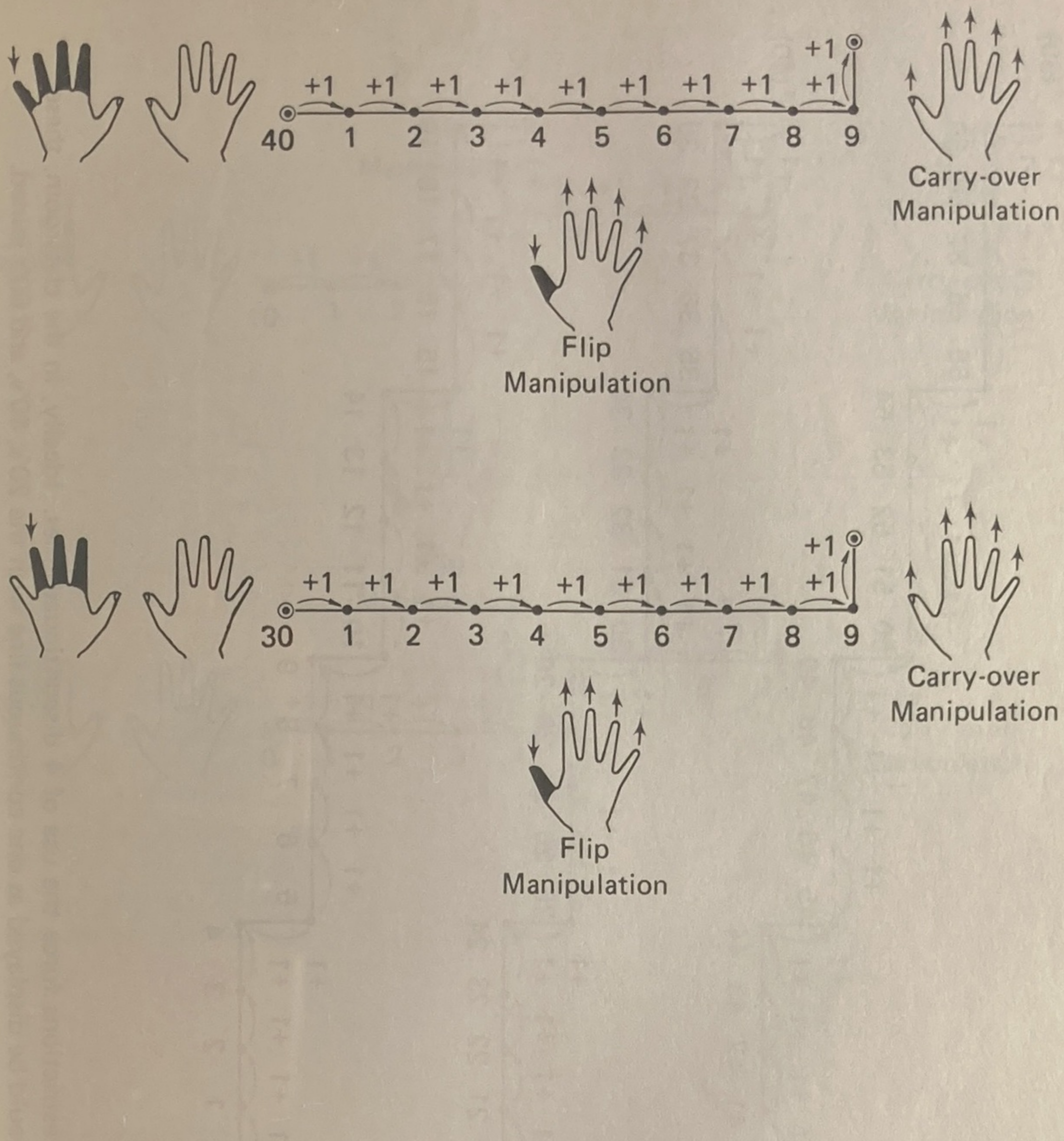
Flip Manipulation



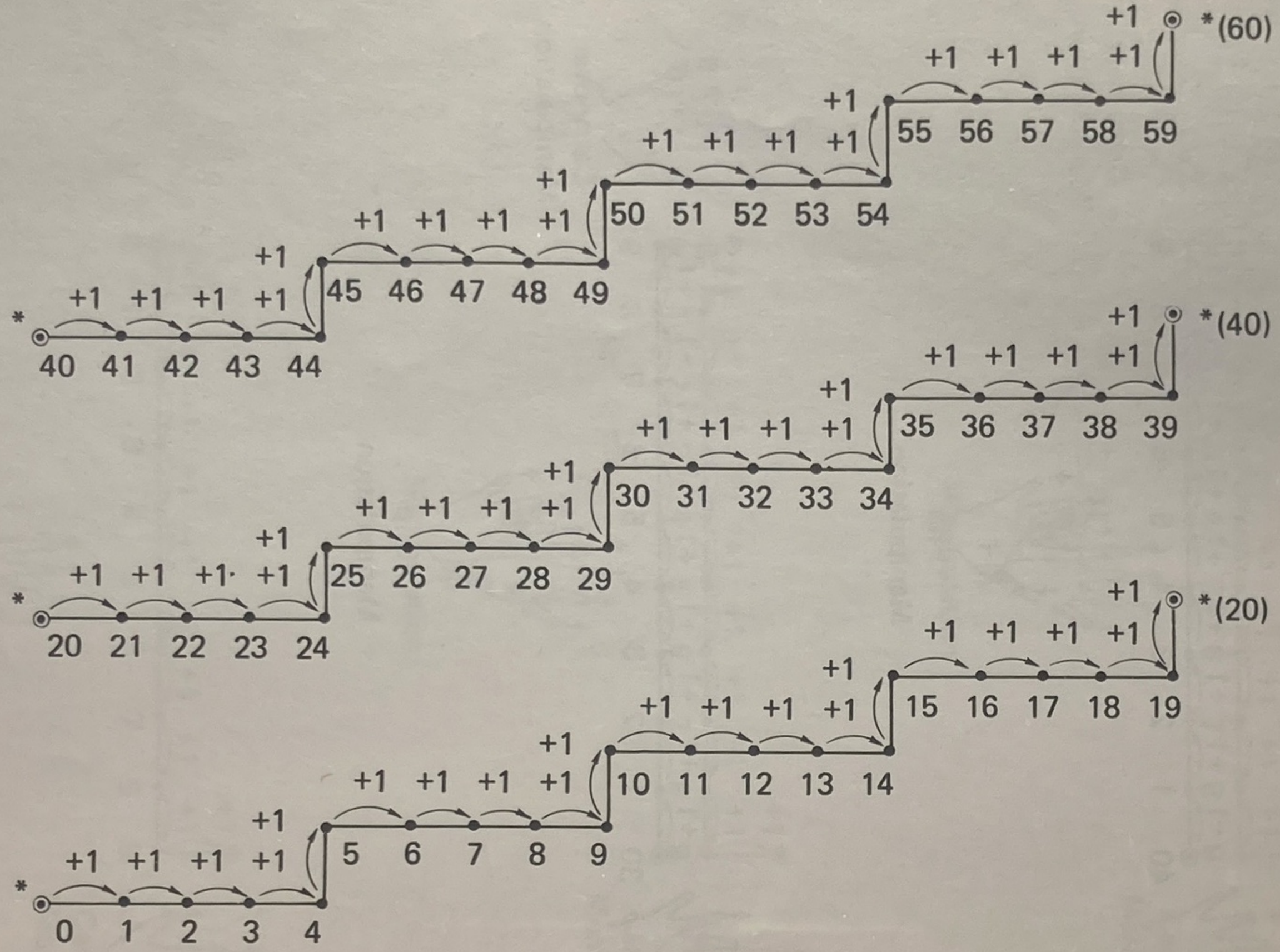

Carry-over Manipulation

Flip Manipulation

# HYPAI DECIMAL NUMBER STAIRS

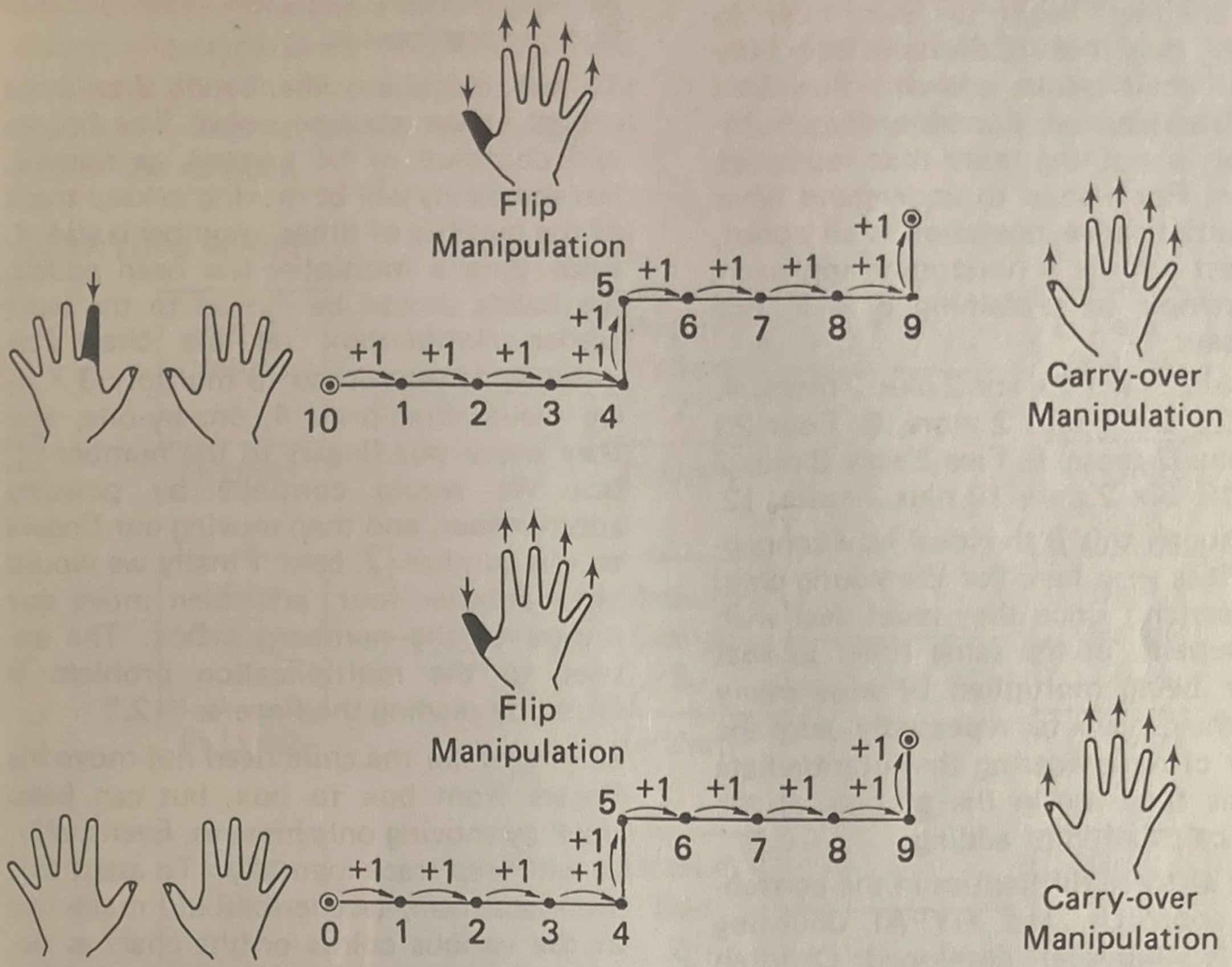


# HYPAI CHISANBOP ™ NUMBER STAIRS




*Space restrictions force the use of 4 discontinued lines. Ideally, in the classroom, these stairs would be displayed as one continuous line with the 20's, 40's, and 60's joined.

# HYPAI CHISANBOP ™ NUMBER STAIRS



## MULTIPLICATION STAGE 1

When children begin to learn how to multiply, they must understand how they achieved their results and why they find the correct answer. Conceptually, multiplication is nothing more than repetitive addition. For a child to understand what this multiplicative operation is all about, it is best to use a horizontal approach. For example in explaining 6 X 2, one would say:

One 2 is 2. Two 2's are 2 plus 2 more, 4. Three 2's are 4 plus 2 more, 6. Four 2's are 6 plus 2 more, 8. Five 2's are 8 plus 2 more, 10. Six 2's are 10 plus 2 more, 12.

Even though this is the ideal way conceptually, it is very hard for the young child to understand since they must deal with four elements at the same time: a) that number being multiplied b) how many times they have to repeatedly add the number c) remembering the intermediate totals as they are in the process d) the mental calculation of adding.

In view of these difficulties in the conventional approach, the HYPAI Counting Aid Chart has been developed. Children can now learn multiplication at a very young age since two of the four elements have been deleted. The number of times the unit has already been added is kept track of by using the boxes on the HYPAI Counting Aid Chart; the intermediate totals are accumulated on the fingers. Thus the child only has to know how to continually add another two and which is the final box (the other multiplier) he is counting up to.

The real value of the HYPAI Counting Aid Chart, besides separating out the two extraneous factors, is the fact that children readily see the meaning of multiplication as repetitive addition. The colors of the HYPAI Counting Aid Chart are deliberately chosen to strengthen the notion of increasing size. By the time that children are ready for multiplication, they already have some perception of the order of numbers, but the Counting Aid Chart strengthens it.

To use the chart, the hands should be placed at the starting point. The fingers will continue to be pressed, as normal, but the hands will be moving to keep track of the number of times a number is added. Each time a multiplier has been added, the hands should be moved to the next higher number-box on the chart. For instance, if we wished to multiply 3 X 4, we would first press 4, one-by-one, and then move our fingers to the number [1] box. We would continue by pressing another four, and then moving our fingers to the number [2] box. Finally we would press another four, and then move our fingers to the number [3] box. The answer to the multiplication problem is found by reading the fingers; "12."

After a while the child need not move his fingers from box to box, but can keep track by moving only his eyes. Eventually, he will keep track mentally. To assist this internalization, teachers should make use of the various colors on the chart as described in the manual. The child will keep track, not only by position, but also by color.

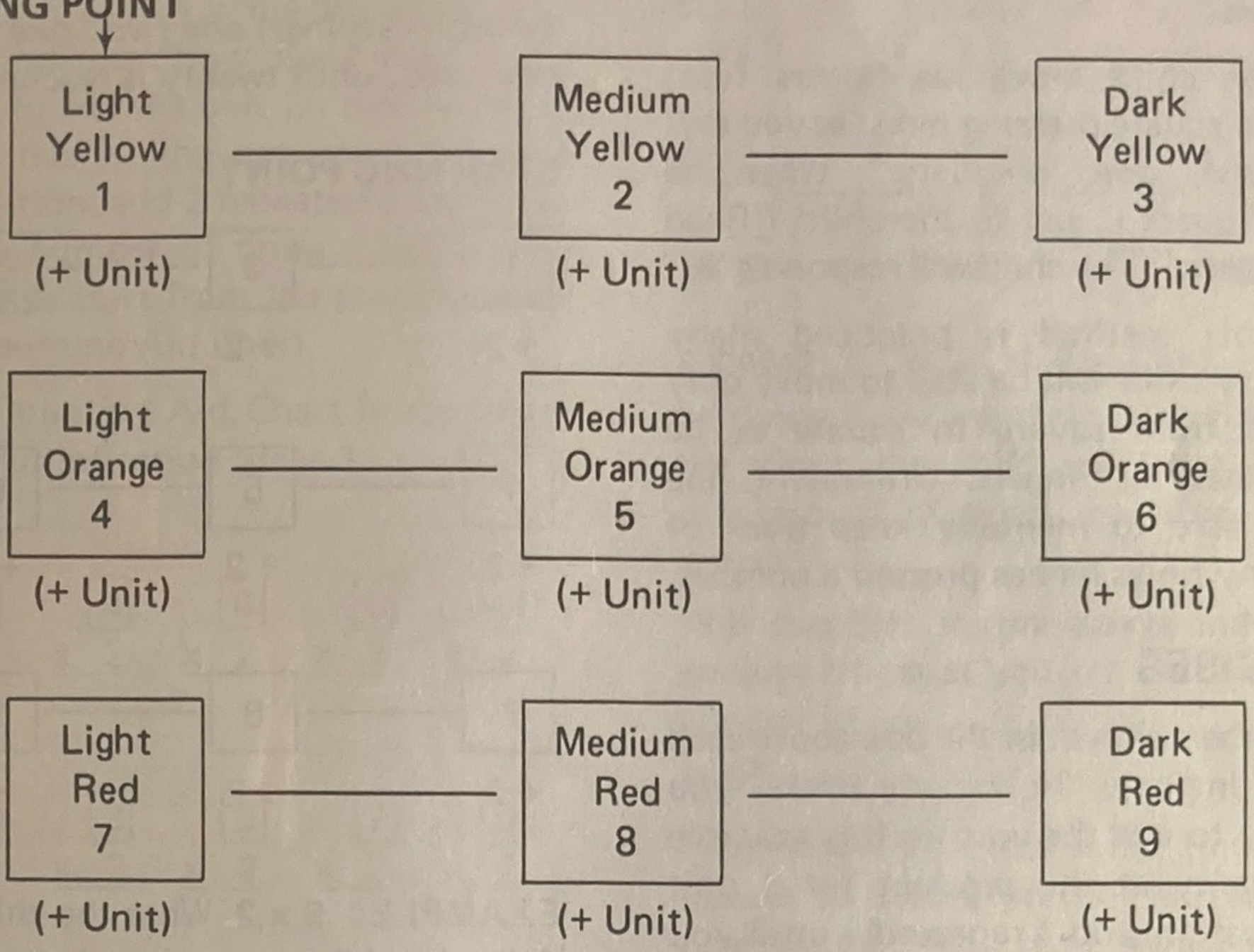
Just as using colors can intensify the lessons, music can also be very helpful with young children and those requiring remedial aid. In this connection, the author is beginning to develop a music and art therapy program which he hopes to integrate into his Chisanbop (TM) theory sometime in the near future.

Although multiplication is a commutative operation, for the young learner, a consistent rule should be established. Therefore, we suggest that in the example 3 X 4 you consistently add the second factor, i.e. you should add 4 three times. If the problem is expressed as $\underline{\times\ 2}^{5}$ it should be read as an instruction to add 2 five times.

## HYPAI COUNTING AID CHART

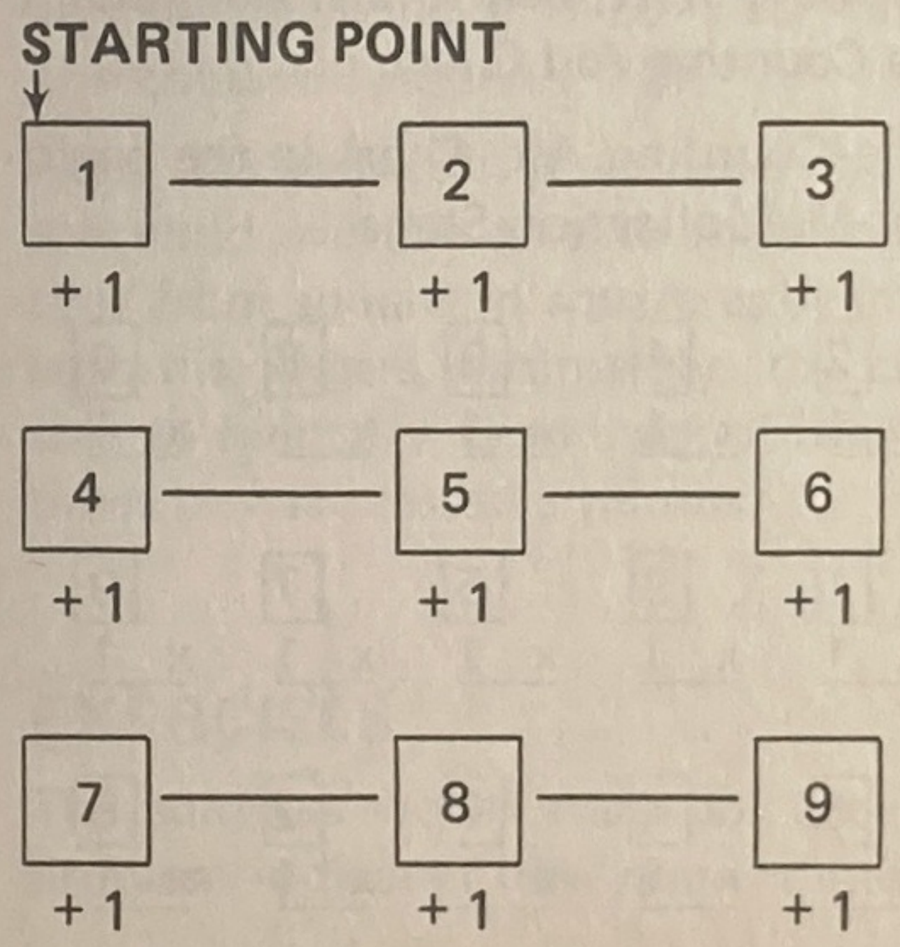
The colors for this chart have been deliberately chosen to darken both horizontally and vertically. The increasing depth of the colors will help to convey the increasing size of the number.

Teachers can construct an enlarged chart and display it prominently in the classroom. Eventually students will be able to locate a box from its position and color.




### ONES

Before using this chart, the child must practice this exercise orally. Press ones to onety (ten)—(saying): *one, one, one, one* ...etc. until onety (ten) is reached.



EXAMPLE: **9 x 1.** What the child will be doing is adding one nine times. With the chart the child can keep count of the amount of times that one is being pressed. The numbers in the boxes indicate the amount of times you are multiplying the number one.

Procedure:

Have the child move his fingers from square to square pressing ones, as you say: *"one, one, one, one*...etc." When he reaches square 9, say to the child, "Read your fingers." The child will respond, "9."

After this method is practiced many times, the child will be able to move only his eyes from square to square as he manipulates his fingers. Ultimately, the child is able to mentally keep track of how many times he has pressed a number.

## EXERCISES

The numbers shown in the box above each problem indicate "how many times" you are going to add the unit (in this example *1*). To arrive at the product for 1 unit multiplication, add 1 repeatedly until you reach the number of times shown in the box. Always start from the starting point on the Counting Aid Chart.

Use the Counting Aid Chart in the beginning of Multiplication Stage 1.

(A) $\boxed{2}\times 1$ $\boxed{4}\times 1$ $\boxed{6}\times 1$ $\boxed{8}\times 1$ $\boxed{0}\times 1$

(B) $\boxed{1}\times 1$ $\boxed{3}\times 1$ $\boxed{5}\times 1$ $\boxed{7}\times 1$ $\boxed{9}\times 1$

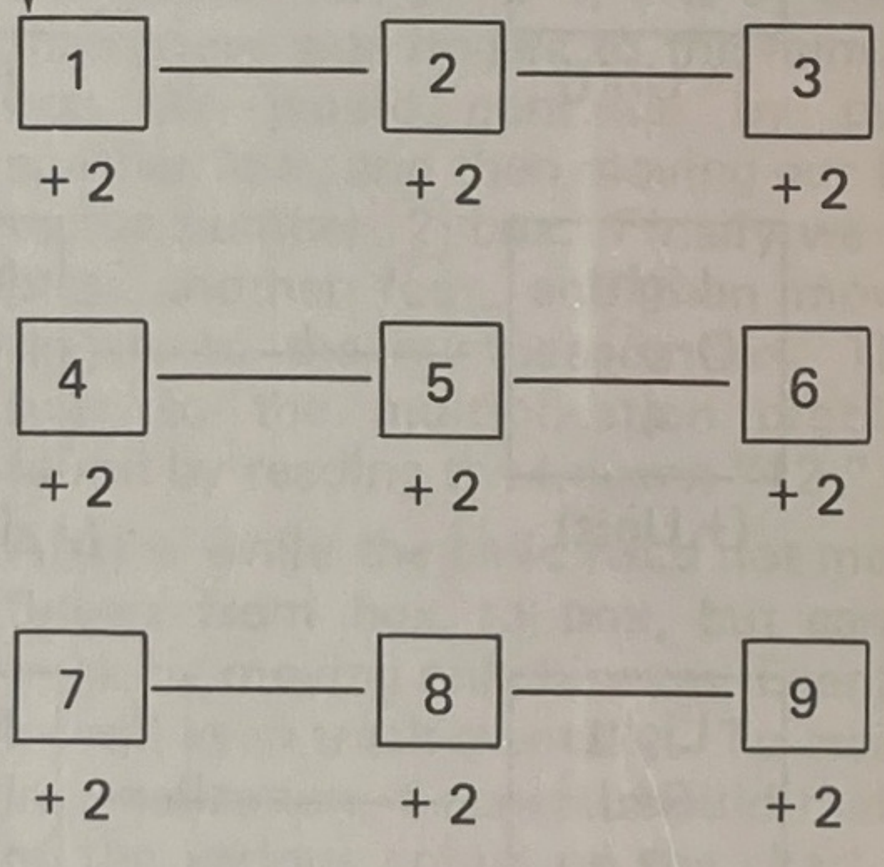
(C) $\boxed{6}\times 1$ $\boxed{4}\times 1$ $\boxed{0}\times 1$ $\boxed{2}\times 1$ $\boxed{8}\times 1$

(D) $\boxed{7}\times 1$ $\boxed{5}\times 1$ $\boxed{9}\times 1$ $\boxed{1}\times 1$ $\boxed{3}\times 1$

## TWOS

Before using this chart, the child must practice this exercise orally. Press twos to twenty—(saying:) *one two...one two...one two...*etc., until twenty is reached.

**STARTING POINT**




EXAMPLE: **9 x 2.** What the child will be doing is adding two nine times. With the chart the child can keep count of the amount of times that two is being pressed.

The numbers in the boxes indicate the amount of times you are multiplying the number two.

Procedure:

Have the child move his fingers from square to square pressing twos, one-by-one, saying: *"one two...one two*...etc." When he reaches square 9, say to the child, "Read your fingers." The child will respond, "18."

After this method is practiced many times, the child will be able to move only his

eyes from square to square as he manipulates his fingers. Ultimately, the child is able to mentally keep track of how many times he has pressed a number.

## EXERCISES

The numbers shown in the box above each problem indicate "how many times" you are going to add the unit (in this example, *2*). To arrive at the product for 2 unit multiplication, add 2 repeatedly until you reach the number of times shown in the box. Always start from the starting point on the Counting Aid Chart.

Use the Counting Aid Chart in the beginning of Multiplication Stage 1.

(A) $\boxed{2}\times 2$ $\boxed{4}\times 2$ $\boxed{6}\times 2$ $\boxed{8}\times 2$ $\boxed{0}\times 2$

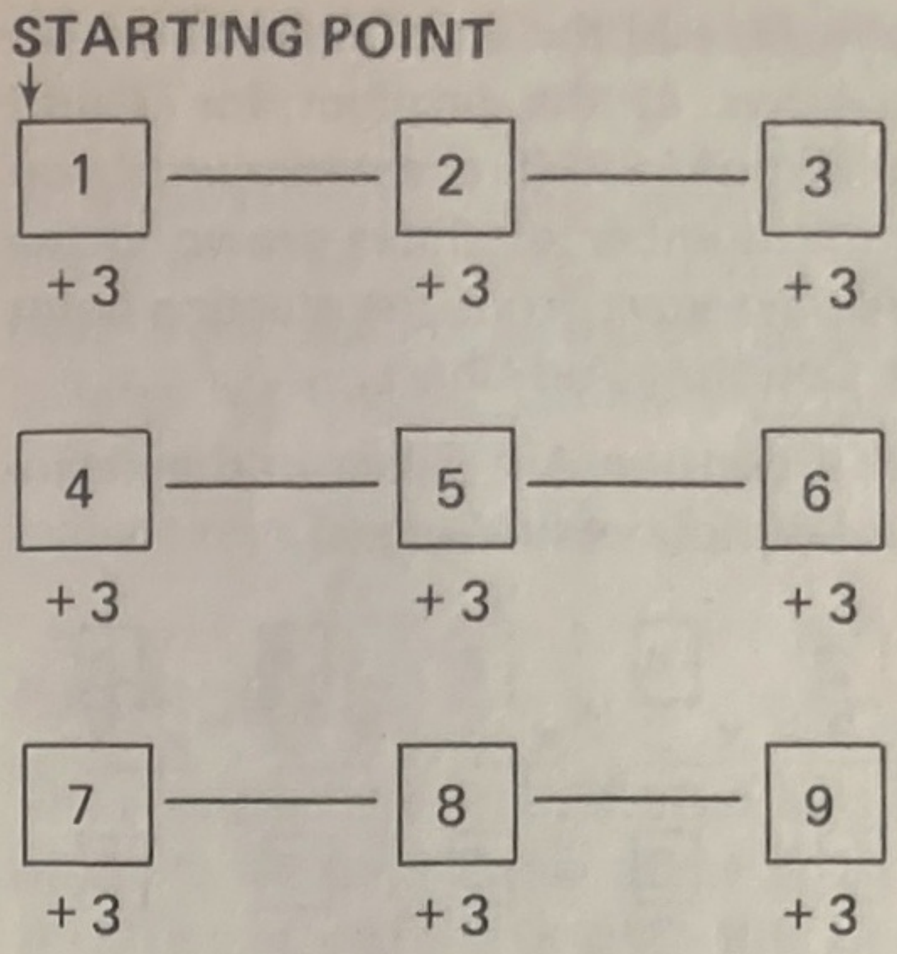
(B) $\boxed{1}\times 2$ $\boxed{3}\times 2$ $\boxed{5}\times 2$ $\boxed{7}\times 2$ $\boxed{9}\times 2$

(C) $\boxed{6}\times 2$ $\boxed{4}\times 2$ $\boxed{0}\times 2$ $\boxed{2}\times 2$ $\boxed{8}\times 2$

(D) $\boxed{7}\times 2$ $\boxed{5}\times 2$ $\boxed{9}\times 2$ $\boxed{1}\times 2$ $\boxed{3}\times 2$

## THREES

Before using this chart, the child must practice this exercise orally. Press threes to thirty—(saying:) *one two three...one two three...*etc., until thirty is reached.




EXAMPLE: **9 x 3.** What the child will be doing is adding three nine times. With the chart the child can keep count of the amount of times that three is being pressed.

The numbers in the boxes indicate the amount of times you are multiplying the number three.

Procedure:

Have the child move his fingers from square to square pressing threes one-by-one saying: *"one two three...one two three...*etc." When he reaches square 9, say to the child, "Read your fingers." The child will respond, "27."

After this method is practiced many times, the child will be able to move only his eyes from square to square as he manipulates his fingers. Ultimately, the child is able to mentally keep track of how many times he has pressed a number.

## EXERCISES

The numbers shown in the box above each problem indicate "how many times" you

are going to add the unit (in this example *3*). To arrive at the product for 3 unit multiplication, add 3 repeatedly until you reach the number of times shown in the box. Always start from the starting point on the Counting Aid Chart.

Use the Counting Aid Chart in the beginning of Multiplication Stage 1.

(A) $\boxed{2}\times 3$ $\boxed{4}\times 3$ $\boxed{6}\times 3$ $\boxed{8}\times 3$ $\boxed{0}\times 3$

(B) $\boxed{1}\times 3$ $\boxed{3}\times 3$ $\boxed{5}\times 3$ $\boxed{7}\times 3$ $\boxed{9}\times 3$

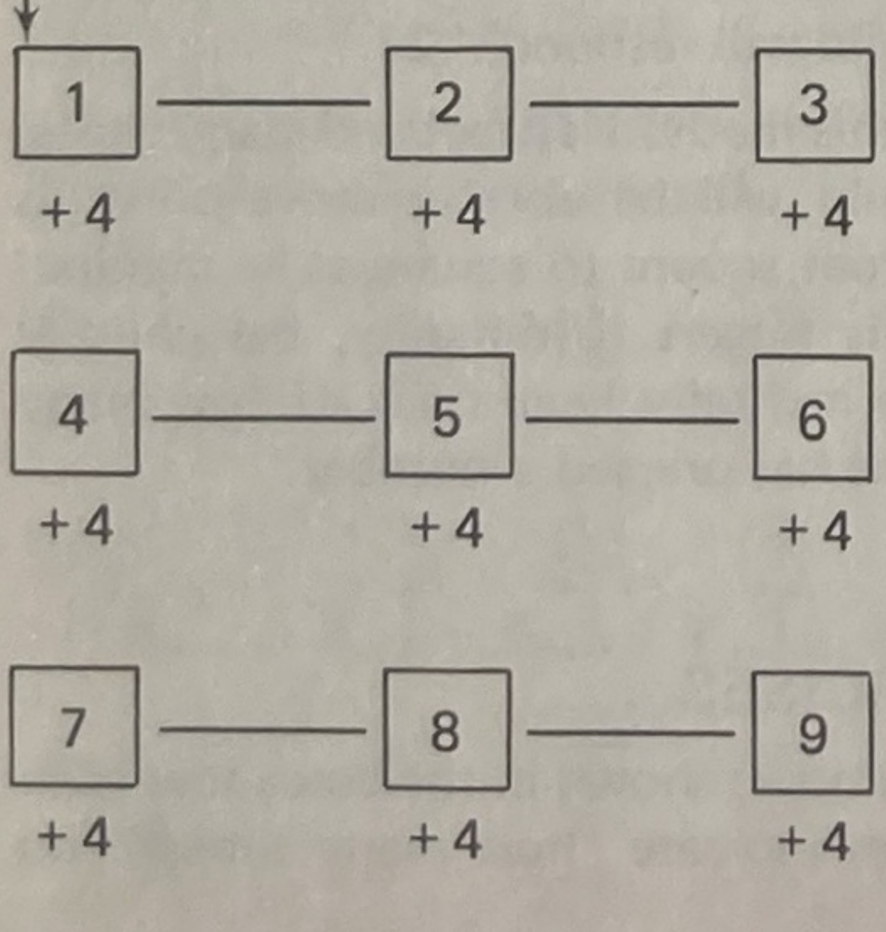
(C) $\boxed{6}\times 3$ $\boxed{4}\times 3$ $\boxed{0}\times 3$ $\boxed{2}\times 3$ $\boxed{8}\times 3$

(D) $\boxed{7}\times 3$ $\boxed{5}\times 3$ $\boxed{9}\times 3$ $\boxed{1}\times 3$ $\boxed{3}\times 3$

### FOURS

Before using this chart, the child must practice this exercise orally. Press fours to forty—(saying:) *one two three four...one two three four...* etc., until forty is reached.

**STARTING POINT**




EXAMPLE: **9 x 4.** What the child will be doing is adding four nine times. With the chart the child can keep count of the amount of times that four is being pressed.

The numbers in the boxes indicate the amount of times you are multiplying the number four.

Procedure:

Have the child move his fingers from square to square pressing fours, one-by-one, saying: *"one two three four...one two three four...* etc." When he reaches square 9, say to the child, "Read your fingers." The child will respond, "36."

After this method is practiced many times, the child will be able to move only his eyes from square to square as he manipulates his fingers. Ultimately, the child is able to mentally keep track of how many times he has pressed a number.

## EXERCISES

The numbers shown in the box above each problem indicate "how many times" you are going to add the unit (in this example, *4*). To arrive at the product for 4 unit multiplication, add 4 repeatedly until you reach the number of times shown in the box.

Use the Counting Aid Chart in the beginning of Multiplication Stage 1.

(A) $\boxed{2}\times 4$ $\boxed{4}\times 4$ $\boxed{6}\times 4$ $\boxed{8}\times 4$ $\boxed{0}\times 4$

(B) $\boxed{1}\times 4$ $\boxed{3}\times 4$ $\boxed{5}\times 4$ $\boxed{7}\times 4$ $\boxed{9}\times 4$

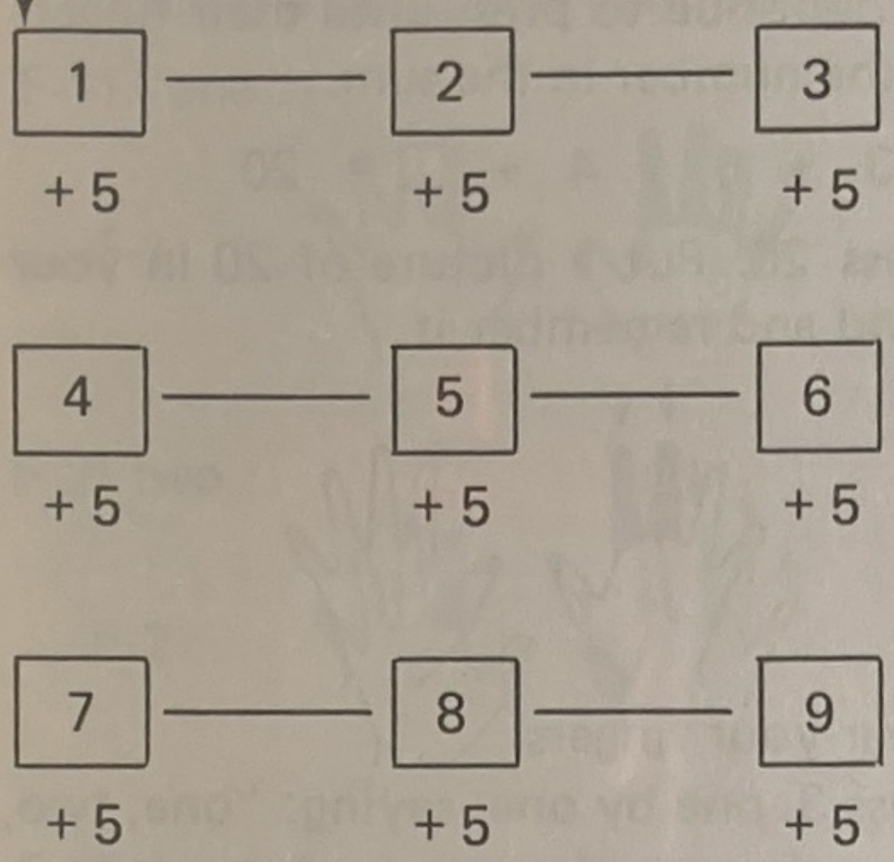
(C) $\boxed{6}\times 4$ $\boxed{4}\times 4$ $\boxed{0}\times 4$ $\boxed{2}\times 4$ $\boxed{8}\times 4$

(D) $\begin{array}{r} \boxed{7} \\ \times\ 4 \\ \hline \end{array}$ $\begin{array}{r} \boxed{5} \\ \times\ 4 \\ \hline \end{array}$ $\begin{array}{r} \boxed{9} \\ \times\ 4 \\ \hline \end{array}$ $\begin{array}{r} \boxed{1} \\ \times\ 4 \\ \hline \end{array}$ $\begin{array}{r} \boxed{3} \\ \times\ 4 \\ \hline \end{array}$

## FIVES

Before using this chart, the child must practice this exercise orally. Press fives to fifty—(saying:) *one, two, three, four, five ...one, two, three, four, five...*etc., until fifty is reached.

**STARTING POINT**




EXAMPLE: **9 x 5.** What the child will be doing is adding five nine times. With the chart the child can keep count of the amount of times that five is being pressed.

The numbers in the boxes indicate the amount of times you are multiplying the number five.

Procedure:

Have the child move his fingers from square to square pressing fives, one-by-one, saying: "*one, two, three, four, five...one, two, three, four, five*...etc." When he reaches square 9, say to the child, "Read your fingers." The child will respond, "45."

After this method is practiced many times, the child will be able to move only his eyes from square to square as he manipulates his fingers. Ultimately, the child is able to mentally keep track of how many times he has pressed a number.

## EXERCISES

The numbers shown in the box above each problem indicate "how many times" you are going to add the unit (in this example, *5*). To arrive at the product for 5 unit multiplication, add 5 repeatedly until you reach the number of times shown in the box.

Use the Counting Aid Chart in the beginning of Multiplication Stage 1.

(A) $\begin{array}{r} \boxed{2} \\ \times\ 5 \\ \hline \end{array}$ $\begin{array}{r} \boxed{4} \\ \times\ 5 \\ \hline \end{array}$ $\begin{array}{r} \boxed{6} \\ \times\ 5 \\ \hline \end{array}$ $\begin{array}{r} \boxed{8} \\ \times\ 5 \\ \hline \end{array}$ $\begin{array}{r} \boxed{0} \\ \times\ 5 \\ \hline \end{array}$

(B) $\begin{array}{r} \boxed{1} \\ \times\ 5 \\ \hline \end{array}$ $\begin{array}{r} \boxed{3} \\ \times\ 5 \\ \hline \end{array}$ $\begin{array}{r} \boxed{5} \\ \times\ 5 \\ \hline \end{array}$ $\begin{array}{r} \boxed{7} \\ \times\ 5 \\ \hline \end{array}$ $\begin{array}{r} \boxed{9} \\ \times\ 5 \\ \hline \end{array}$

(C) $\begin{array}{r} \boxed{6} \\ \times\ 5 \\ \hline \end{array}$ $\begin{array}{r} \boxed{4} \\ \times\ 5 \\ \hline \end{array}$ $\begin{array}{r} \boxed{0} \\ \times\ 5 \\ \hline \end{array}$ $\begin{array}{r} \boxed{2} \\ \times\ 5 \\ \hline \end{array}$ $\begin{array}{r} \boxed{8} \\ \times\ 5 \\ \hline \end{array}$

(D) $\begin{array}{r} \boxed{7} \\ \times\ 5 \\ \hline \end{array}$ $\begin{array}{r} \boxed{5} \\ \times\ 5 \\ \hline \end{array}$ $\begin{array}{r} \boxed{9} \\ \times\ 5 \\ \hline \end{array}$ $\begin{array}{r} \boxed{1} \\ \times\ 5 \\ \hline \end{array}$ $\begin{array}{r} \boxed{3} \\ \times\ 5 \\ \hline \end{array}$

This is the last Counting Aid Chart that is presented. By now most students will be ready for second stage addition and multiplication. However, teachers can construct similar charts for the numbers 6, 7, 8, and 9.

## ALGEBRA–(MISSING ADDENDS) CALCULATIONS–STAGE 1

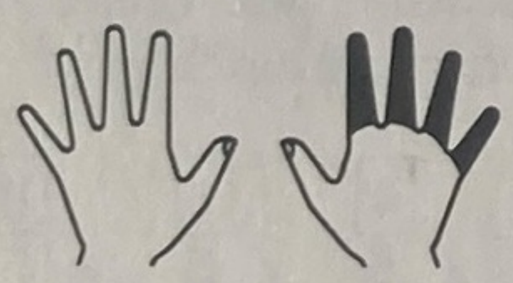
One of the concepts that a youngster can grasp with remarkable ease through the use of Chisanbop ™ is that of the missing addend. He has only to manipulate his fingers and be able to count to the picture of the sum in order to achieve the correct answer. This is primarily an oral and visual process.

Below is the exact procedure to be followed.

$$1 + \square = 4$$

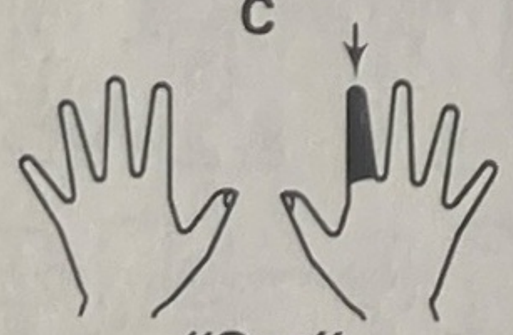
A. Press the sum and put a picture of that sum in your mind (in this example, 4). Remember that picture!

A



B. Clear your fingers.
C. Press the first addend.

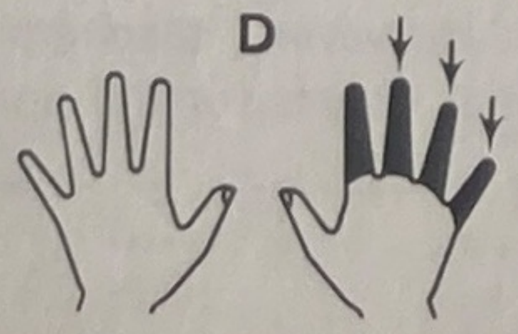
C



*"One"*

D. While keeping the first addend pressed, press, one-by-one, until your fingers reach the picture you are remembering. Make sure to count as you press.

D



*"One, two, three"*

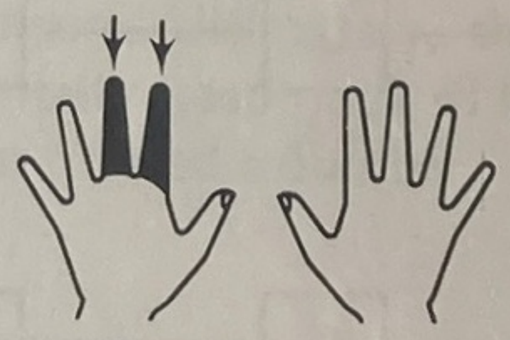
**Note:** The visual sense is used in making sure the pressed fingers are the same in steps A & D. The *oral counting* determines what is the missing addend. The procedure is extremely beneficial in so far as the missing addend is found by *adding* not as in some textbooks which turn the problem into one of subtraction.

Once the students are familiar with the concept of missing addends, they can skip steps A and B. They should press the known addend, and then counting as they press, continue to press until their fingers show the number in the sum.

$$3 + 6 + 4 + \square = 20$$

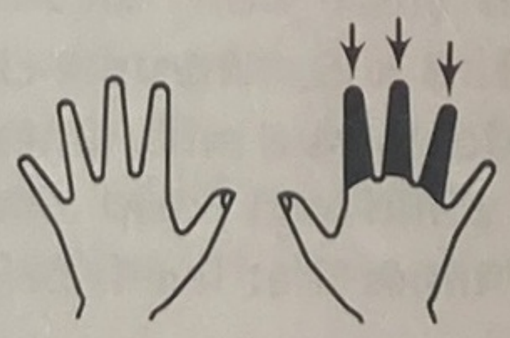
A) Press 20. Put a picture of 20 in your mind and remember it.

A



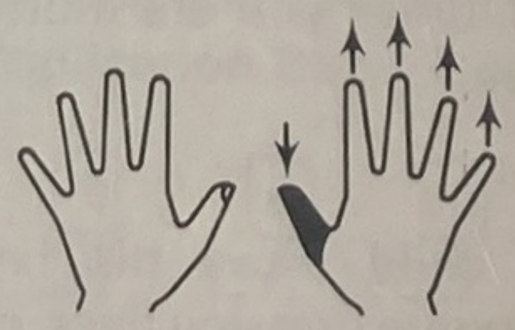
B) Clear your fingers.
C) Press 3, one by one, saying: "one, two, three."

C



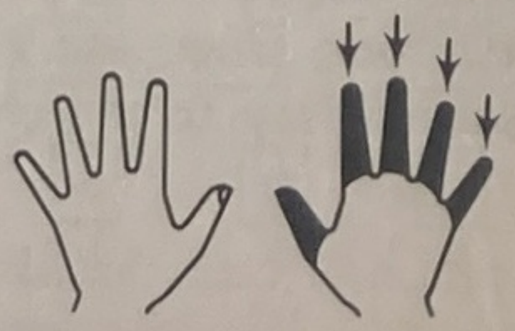
D-1) Press six more saying "one, two

D-1



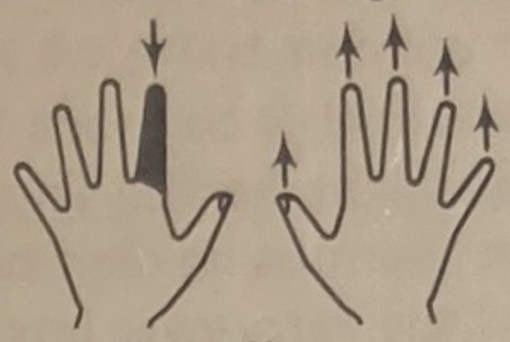
D-2) three, four, five, six."

D-2



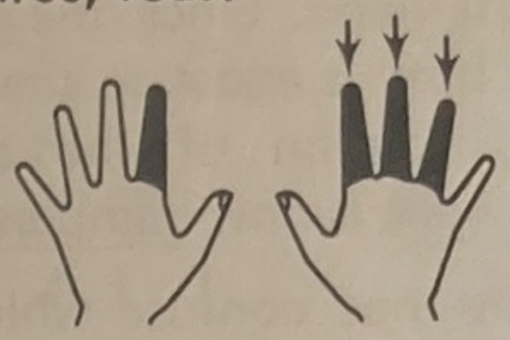
E-1) Press four more saying: "one

E-1



E-2) two, three, four."

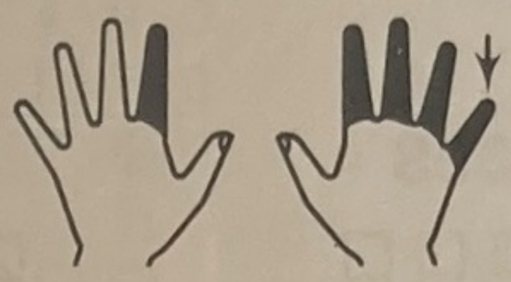
E-2



F) Count and press, one-by-one, until you reach the picture in your mind.

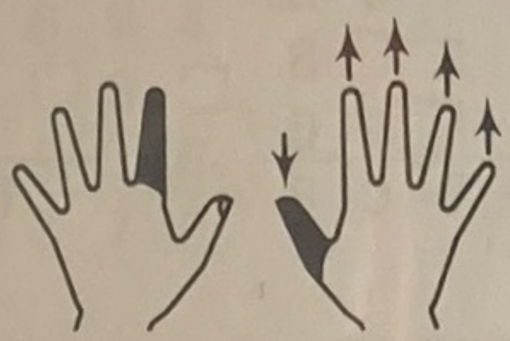
F-1) "one

F-1



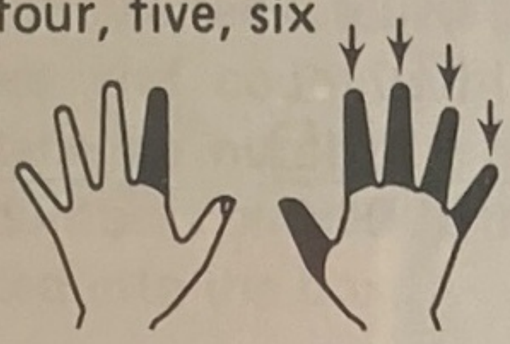
F-2) two

F-2



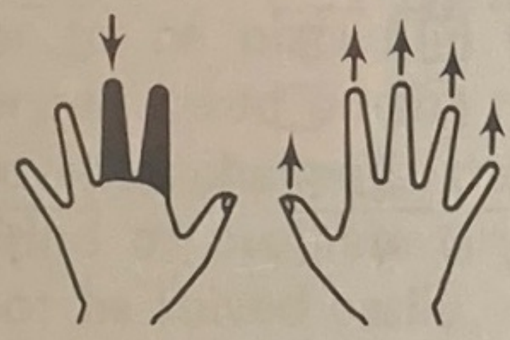
F-3) three, four, five, six

F-3



F-4) seven."

F-4



The missing addend is the number you counted.

As the students first learn this operation, it is helpful to give many problems where the sum is identical. They can concentrate on learning the new operation without worrying about what the number is they are pressing to.

## EXERCISES

(A)

| | |
|---|---|
| 1 + □ = 4 | 1 + □ = 5 |
| 2 + □ = 4 | 2 + □ = 5 |
| 3 + □ = 4 | 3 + □ = 5 |
| 4 + □ = 4 | 4 + □ = 5 |

(B)

| | |
|---|---|
| 1 + □ = 9 | 5 + □ = 9 |
| 2 + □ = 9 | 6 + □ = 9 |
| 3 + □ = 9 | 7 + □ = 9 |
| 4 + □ = 9 | 8 + □ = 9 |

(C)

| | |
|---|---|
| 1 + □ = 10 | 5 + □ = 10 |
| 2 + □ = 10 | 6 + □ = 10 |
| 3 + □ = 10 | 7 + □ = 10 |
| 4 + □ = 10 | 8 + □ = 10 |

(D)

| | |
|---|---|
| 10 + □ = 15 | 8 + □ = 13 |
| 11 + □ = 15 | 9 + □ = 15 |
| 12 + □ = 15 | 7 + □ = 16 |
| 13 + □ = 15 | 5 + □ = 12 |

Once students have learned the basics of this operation, teachers should begin to vary the number in the sum. At this point students become more interested in variety than in the security they had in always pressing the same sum.

## EXERCISES

(A)

| | |
|---|---|
| □ + 6 = 11 | □ + 2 = 11 |
| □ + 7 = 12 | □ + 5 = 14 |
| □ + 8 = 13 | □ + 4 = 13 |
| □ + 9 = 14 | □ + 3 = 12 |

(B)

| | |
|---|---|
| □ + 16 = 20 | □ + 12 = 17 |
| □ + 17 = 24 | □ + 13 = 20 |
| □ + 18 = 22 | □ + 14 = 19 |
| □ + 19 = 25 | □ + 15 = 24 |

(C)

| | |
|---|---|
| □ + 19 = 24 | □ + 3 + 6 = 14 |
| □ + 28 = 32 | □ + 9 + 1 = 12 |
| □ + 24 = 30 | □ + 5 + 4 = 18 |
| □ + 20 = 29 | □ + 3 + 2 = 13 |

(D)

| | |
|---|---|
| 7 + □ + 7 = 21 | 4 + 6 + □ = 14 |
| 4 + □ + 8 = 16 | 5 + 9 + □ = 22 |
| 5 + □ + 6 = 15 | 3 + 8 + □ = 18 |
| 3 + □ + 9 = 19 | 7 + 8 + □ = 23 |

## MULTIPLE ADDENDS– MISSING DIGITS

As a comprehensive test in addition—the nature of addition, the concept of carry-over, and the meaning of missing addends—an exercise such as the one below is very helpful. This type of problem also excites older children when they realize they are performing problems most adults would find impossible.

$$\begin{array}{r} \square\,\square\,\square\,\square \\ 2,\square\,7\;\;8 \\ 4,\;9\;\square\,2 \\ 6,\;5\;\;8\;\square \\ +\;\square,1\;\;5\;\;6 \\ \hline 2\;\;3,0\;\;2\;\;3 \end{array}$$

The process of finding these missing digits is really quite easy. The known digits in the units' column must first be added. Their total is 16. The next step is to continue to add until the *right* hand shows a picture of three. Having counted up to 7 to reach the "picture of 3," 7 is placed in the box. Since there is 20 registered on the left hand, a 2 is placed in the carry-over box.

When we add the known digits in the tens' column, we reach a total of 22. Since the tens' digit in the answer is 2, and since our right hand reads 2, we know that nothing more has to be added. Consequently, we place a 0 in the missing digit box, and a 2 in the carry-over box.

After adding the known digits of the hundreds' column, our fingers read 17. We must count out an additional 3 so that our right hand shows the 0 that is in the answer. Since we counted out 3, a 3 is placed in the missing digit box. Since our left hand now reads 20, we place a 2 in the carry-over box.

Since the total of the known digits in the thousands' column is 14, we must count out an additional 9 to reach the picture of 3 on our right hands. Therefore, a 9 is placed in the missing digit box, and a 2 is placed in the carry-over box after reading our left hand. Since the two in the carry-over box, is equal to the two in the thousands' column of the answer, we know our work is now complete.

Caution: Do not confuse which number goes where. The number you have counted is the missing digit. The number on the left hand goes into the carry-over box.

## EXERCISES

(A)

$$\begin{array}{r} \square\,\square\,\square \\ \;\;1\;6\;\square \\ \;\;7\;\square\,8 \\ +\;\;\square\,9\;6 \\ \hline 1,5\;0\;8 \end{array} \qquad \begin{array}{r} \square\,\square\,\square \\ \;\;\square\,7\;5 \\ \;\;6\;\square\,5 \\ +\;\;4\;3\;\square \\ \hline 1,6\;0\;0 \end{array}$$

$$\begin{array}{r} \square\,\square\,\square \\ \;\;4\;\square\,7 \\ \;\;\square\,7\;3 \\ +\;\;5\;0\;\square \\ \hline 1,0\;9\;4 \end{array} \qquad \begin{array}{r} \square\,\square\,\square \\ \;\;1\;8\;\square \\ \;\;\square\,0\;9 \\ +\;\;7\;\square\,5 \\ \hline 1,6\;0\;5 \end{array}$$

(B)

$$\begin{array}{r} \square\,\square\,\square \\ \;\;\square\,4\;3 \\ \;\;5\;7\;\square \\ +\;\;1\;\square\,9 \\ \hline 1,0\;4\;2 \end{array} \qquad \begin{array}{r} \square\,\square\,\square \\ \;\;5\;\square\,0 \\ \;\;\square\,8\;9 \\ +\;\;7\;3\;\square \\ \hline 1,5\;6\;8 \end{array}$$

$$\begin{array}{r} \square\,\square\,\square \\ \;\;2\;4\;\square \\ \;\;9\;\square\,3 \\ +\;\;\square\,8\;0 \\ \hline 1,9\;5\;0 \end{array} \qquad \begin{array}{r} \square\,\square\,\square \\ \;\;\square\,\square\,\square \\ \;\;2\;4\;8 \\ +\;\;9\;0\;5 \\ \hline 1,8\;0\;0 \end{array}$$

(C)

```
 □□□          □□□
  699          293
  □□□          410
+ 950        + □□□
2,149        1,247
```

```
 □□□          □□□
  □3□          126
  649          □□8
+ 7□2        + 94□
2,210        1,575
```

(D)

```
 □□□          □□□
  7□□          563
  490          □□4
+ □70        + 29□
1,750        1,512
```

```
 □□□          □□□
  694          1□3
  □8□          926
+ 9□5        + □8□
2,379        1,432
```

If the missing digit is in an addend, the number to be inserted is found by continuing to press and count until the right hand registers the number shown in the sum. The number *counted* is the number to be inserted into the box.

If the missing digit is found in the sum, the number to be placed in the box is the number *registered on the right hand.* Students love to challenge their parents with this kind of problem since the answer can not be solved easily, even while using an electronic calculator.

## EXERCISES

(A)

```
 □□□          □□□
  □62          759
  743          □□6
+ 59□        + 983
1,9□3        2,50□
```

```
 □□□          □□□
  2□4          736
  93□          1□3
+ 658        + 954
□,□75        □,□1□
```

(B)

```
 □□□          □□□
  975          846
  3□□          739
+ 610        + □2□
□,□28        2,0□5
```

```
 □□□          □□□
  □□2          46
  436          529
+ 815        + 831
2,17□        1,6□□
```

(C)

```
 □□□          □□□
  65□          2□8
  9□3          836
+ □10        + □75
2,063        1,52□
```

```
 □□□          □□□
  874          75□
  532          □42
+ 9□6        + 513
□,□4□        1,9□3
```

## ADDITION STAGE 2

Addition Stage 2 has been designed for use by Chisanbop ™ students who have successfully completed Addition Stage 1, the entire contents of which required manipulations using unit by unit counting exclusively.

Stage 2 introduces the first manipulations for increasing speed. In truth, once Stage 2 is thoroughly understood, the speed of the practitioner should be at least twice as quick as in Stage 1. Stage 2 centers around the number 5. In fact the smarter students may already have begun to use some Stage 2 maneuvers. *The numbers 1 through 4 are pressed exactly the same way they were pressed in Stage 1, unit by unit. The number 5 is pressed directly, either by pressing the right thumb if it is available for pressing, or by exchanging 5 (10–5), that is pressing the first available ten and clearing 5. The numbers 6 through 9 are pressed in a combination of maneuvers to be shown below.*

On recognition of the number 5 or of higher numbers (6, 7, 8, 9), the students will first PRESS 5 or EXCHANGE 5 (PRESS 10, CLEAR 5) and then press their fingers one-by-one to complete the rest of the number. The students must learn to recognize whether the Fifth Finger is AVAILABLE or NOT AVAILABLE respectively allowing a direct Press of 5 or requiring an Exchange 5 (Press 10, Clear 5). Stage 2 must be understood completely before moving on to Stage 3. Each stage builds upon the earlier stages. Stage 2 will continue to give students insights into mathematics, e.g. the renaming of numbers. It deliberately eases the students into the various manipulations needed for speed but does it step-by-step.

### How to Press the Units— Stage 2

*1-4* The numbers 1-4 are pressed as in Stage 1, unit by unit.

*5* A) If the right thumb is available for pressing, 5 can be added by merely pressing the right thumb.

B) If the right thumb is already pressed, 5 is added by pressing the first available 10 and clearing the thumb on the right hand. This second manipulation is called "Exchange 5."

*6-9* The numbers 6 through 9 are pressed A) first by pressing 5 (either directly or through Exchange 5) and then by pressing the rest of the number unit by unit, (as in Stage 1) saying *six, seven, eight, nine.*

It is sometimes helpful, especially for younger children at the very beginning of Addition Stage 2, to verbalize the unit value being pressed. Thus the number eight would be pressed as "five, and *ONE* more, six, and *ONE* more, seven, and *ONE* more, eight."

Very rapidly the verbal instruction "and *ONE* more" may be deleted. For children who have not as yet internalized their number facts this method of pressing is essential since they do not yet know that 8 = 5 + 3. These manipulations will, however, enable them to memorize these number facts quicker than usual.

## HOW TO READ NUMBERS (ADDENDS)

| STAGE 1 ADDENDS | BREAKDOWN OF ADDENDS | SAYING: |
|---|---|---|
| 1 | 1 | *ONE* |
| 2 | 1 + 1 | *ONE, TWO* |
| 3 | 1 + 1 + 1 | *ONE, TWO, THREE* |
| 4 | 1 + 1 + 1 + 1 | *ONE, TWO, THREE, FOUR* |
| **1–4** | **ADD UNIT BY UNIT** | **SAYING:** *ONE, TWO, THREE, FOUR* |

| STAGE 2 ADDENDS | BREAKDOWN OF ADDENDS | SAYING: |
|---|---|---|
| 5 | 5 | *PRESS FIVE* or *EXCHANGE FIVE* |
| 6 | 5 + 1 | *PRESS FIVE* or *EXCHANGE FIVE* THEN *SIX* |
| 7 | 5 + 1 + 1 | *PRESS FIVE* or *EXCHANGE FIVE* THEN *SIX, SEVEN* |
| 8 | 5 + 1 + 1 + 1 | *PRESS FIVE* or *EXCHANGE FIVE* THEN *SIX, SEVEN, EIGHT* |
| 9 | 5 + 1 + 1 + 1 + 1 | *PRESS FIVE* or *EXCHANGE FIVE* THEN *SIX, SEVEN, EIGHT, NINE* |
| **5–9** | **ADD 5 FIRST THRU EITHER ONE OF:** PRESS 5TH FINGER EXCHANGE 5(10–5) THEN ADD UNIT BY UNIT. | **SAYING: EITHER** *PRESS FIVE* or *EXCHANGE FIVE* THEN *SIX, SEVEN, EIGHT, NINE* |

When the 5th finger is AVAILABLE FOR PRESSING, press the 5th finger directly to add 5. When the 5th finger is NOT AVAILABLE FOR PRESSING (Exchange 5) by pressing 10 and clearing 5 simultaneously.

## Six Key Manipulations

There are in Stage 2, six key manipulations (see illustrations). Once these key manipulations are learned and automatic, the student will have completed Additon Stage 2. The first two key manipulations are those from Stage 1, 4 + 1 and 9 + 1. The second pair of manipulations are learned readily: 4 + 5 and 9 + 5. The last pair of manipulations are the most difficult, especially for adults, they are 4 + 6 and 9 + 6.

If you think of the value of the number as you make the two-step manipulation in these latter moves (plus 5 and then plus 1) you will be able to master them more quickly.

These are Six Calculations of Addition that represent Key Manipulations for the student and that must be recognized and corrected by the teacher. In each, the student tends to execute an incorrect finger exchange because the value of fifth finger has not been firmly established.

## Big Exchange 5

To add 5, when there is exactly 40 pressed on the left hand and 5 or more pressed on the right hand a "Big Exchange 5" must be made. (See illustration.)

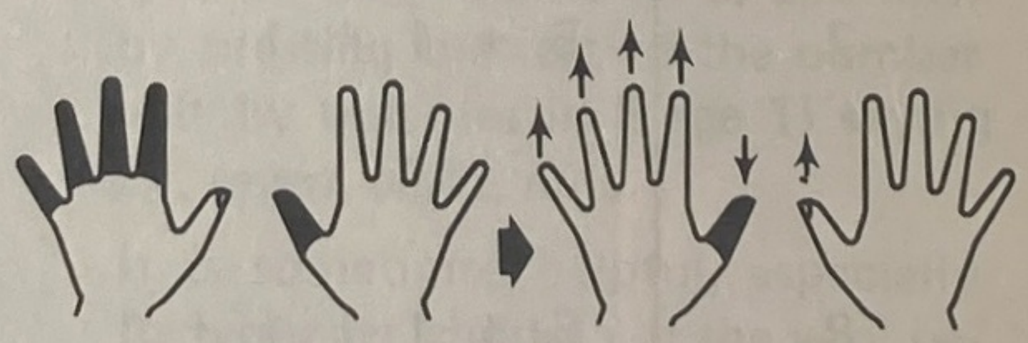
The *Big Exchange 5* is performed by pressing the left thumb, clearing all the other fingers on the left hand, and clearing the right thumb all simultaneously.

In terms of number value, it means adding 50 (pressing the left thumb), subtracting 40 (clearing the other fingers of the left hand), and subtracting another 5 (clearing the right thumb).

If the numbers between 6-9 are to be added in this situation, a Big Exchange 5 would be made and the rest of the number would be added, one-by-one.

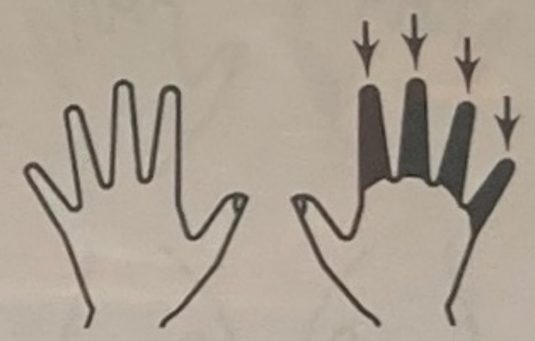
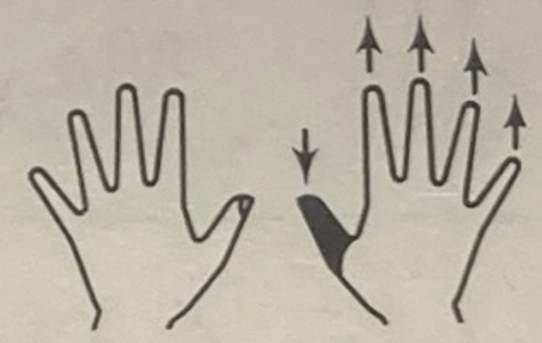
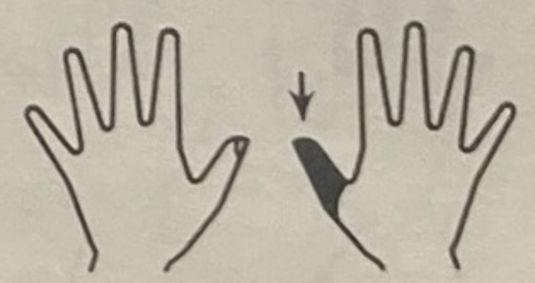
**Big Exchange 5**

45 + 5



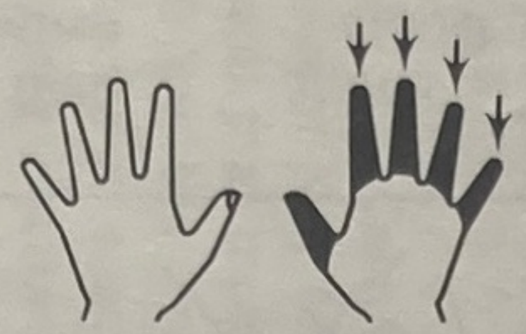
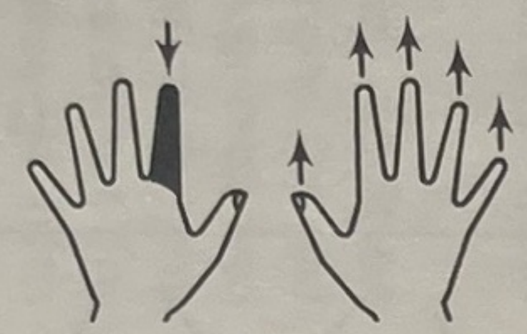
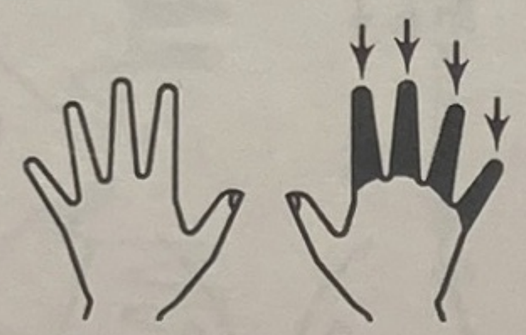
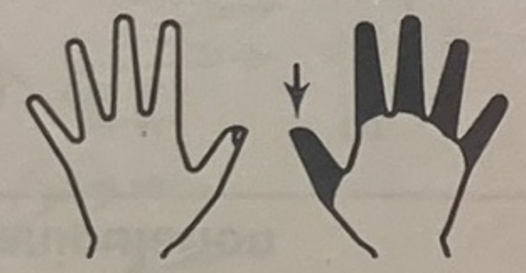
Simultaneously
Left hand—press 50,
clear 40
Right hand—Clear 5

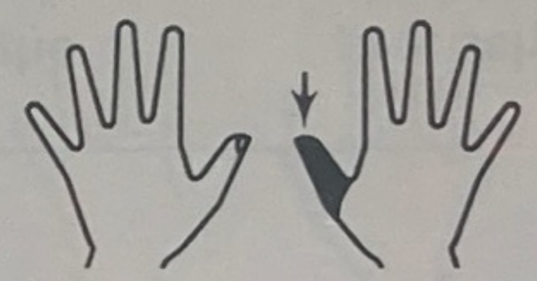
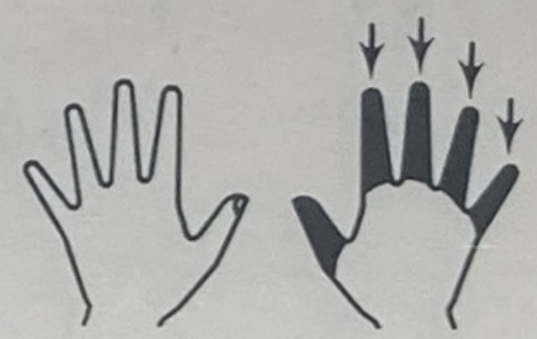
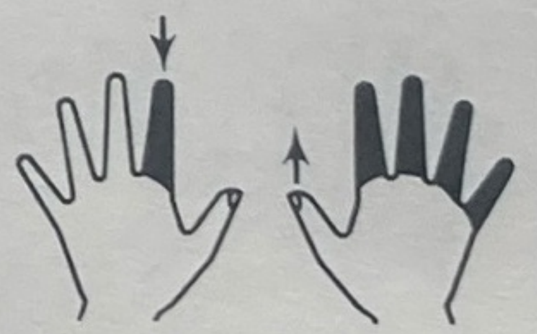
## SIX KEY MANIPULATIONS OF STAGE 2

| Example | Stage | Breakdown of No. | Manipulation | Verbal Count* | Voice Intonation |
|---|---|---|---|---|---|
| 1)<br>A) **4** | A) 1st | A) 1 + 1 + 1 + 1 | A)  | A) one (1), two (1), three (1), four (1) | A) Average |
| B) <u>+1</u> | B) 1st | B) 1 | B)  | B) one (1) | B) Higher** |
| 2)<br>A) **9** | A-1) 2nd | A-1) 5 | A-1)  | A-1) Press five (5) | A-1) Average |

*Verbal Count: The printed number indicates the number to be spoken. The number in the parenthesis indicates the value added with the manipulation. It is sometimes helpful, at the beginning to indicate precisely what value is being added with each individual manipulation. Thus nine would be expressed Five (5), and *ONE* more, six, and *ONE* more, seven, and *ONE* more, eight, and *ONE* more nine.

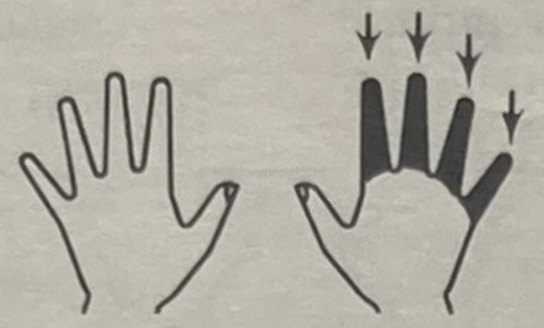
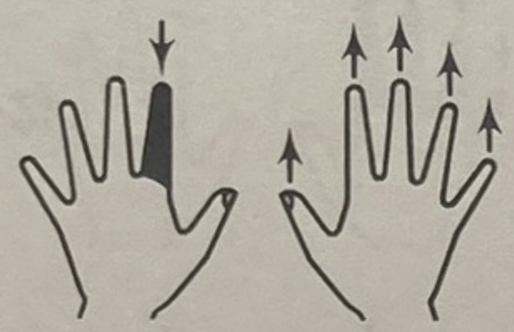
**Inflect the voice at key manipulations.

| Example | Stage | Breakdown of No. | Manipulation | Verbal Count* | Voice Intonation |
|---|---|---|---|---|---|
| | A-2) 1st | A-2) 1 + 1 + 1 + 1 | A-2)  | A-2) six (1), seven (1), eight (1), nine (1) | A-2) Average |
| B) <u>+1</u> | B) 1st | B) 1 | B)  | B) one (1) | B) Higher |
| 3) | | | | | |
| A) 4 | A) 1st | A) 1 + 1 + 1 + 1 | A)  | A) one (1), two (1), three (1), four (1) | A) Average |
| B) <u>+5</u> | B) 2nd | B) 5 | B)  | B) Press five (5) | B) Lower |

| Example | Stage | Breakdown of No. | Manipulation | Verbal Count* | Voice Intonation |
|---|---|---|---|---|---|
| 4) A) 9 | A-1) 2nd | A-1) 5 | A-1)  | A-1) Press five (5) | A-1) Average |
| | A-2) 1st | A-2) 1 + 1 + 1 + 1 | A-2)  | A-2) six (1), seven (1), eight (1), nine (1) | A-2) Average |
| B) +5 | B) 2nd | B) 5 | B)  | B) Exchange five (10-5) | B) Lower |

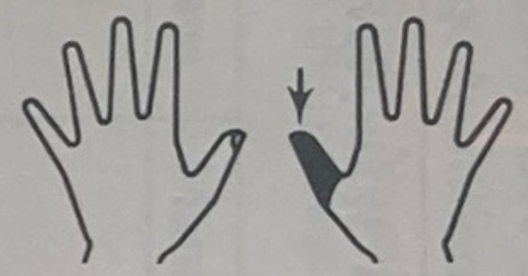
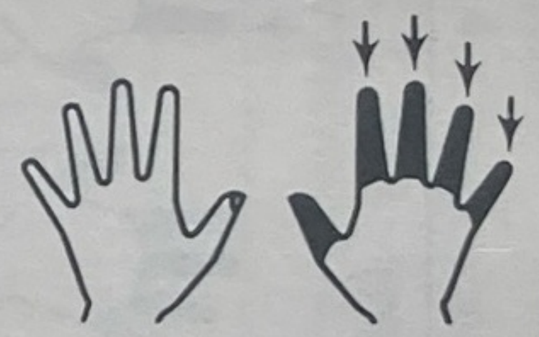
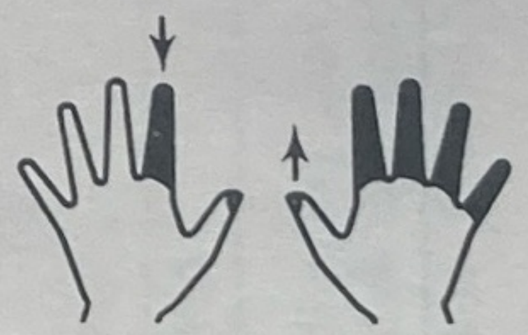
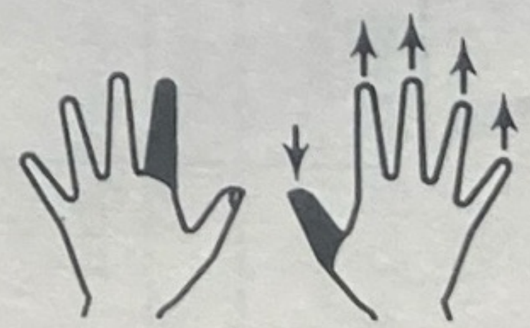
*Verbal Count: The printed number indicates the number to be spoken. The number in the parenthesis indicates the value added with the manipulation. It is sometimes helpful, at the beginning to indicate precisely what value is being added with each individual manipulation. Thus six would be expressed Five (5), and *ONE* more, six.

**Inflect the voice at key manipulations.

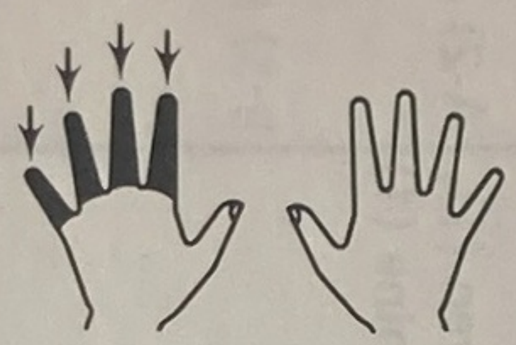
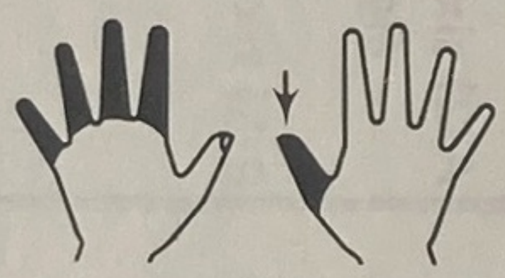
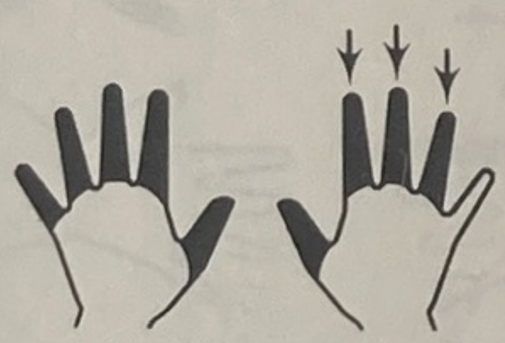
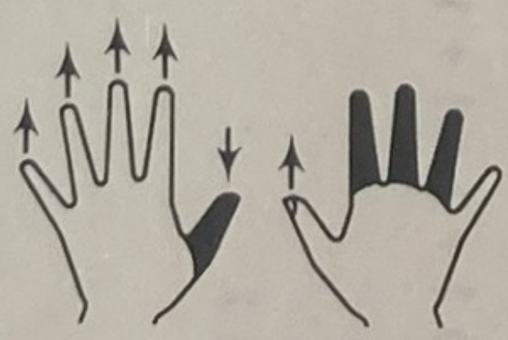
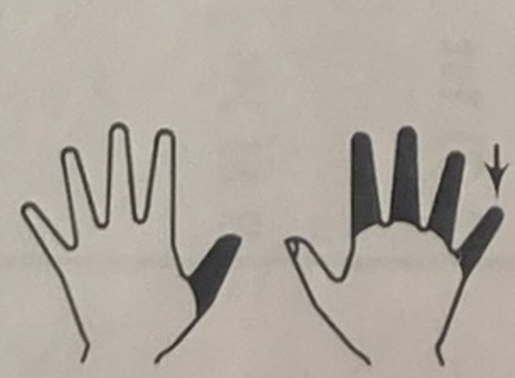
| Example | Stage | Breakdown of No. | Manipulation | Verbal Count* | Voice Intonation |
|---|---|---|---|---|---|
| 5)<br>A) **4** | A) 1st | A) 1 + 1 + 1 + 1 | A)  | A) one (1), two (1), three (1), four (1) | A) Average |
| B) **+6** | B-1) 2nd | B-1) 5 | B-1) | B-1) Press five (5) | B-1) Lower |
| | B-2) 1st | B-2) 1 | B-2)  | B-2) six (1) | B-2) Higher |

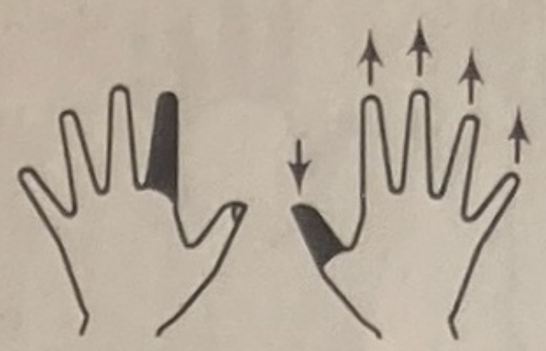
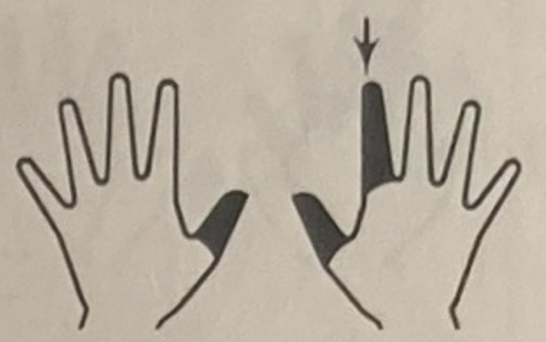
*Verbal Count: The printed number indicates the number to be spoken. The number in the parenthesis indicates the value added with the manipulation. It is sometimes helpful, at the beginning to indicate precisely what value is being added with each individual manipulation. Thus seven would be expressed Five (5), and *ONE* more, six, and *ONE* more, seven.

**Inflect the voice at key manipulations.

| Example | Stage | Breakdown of No. | Manipulation | Verbal Count* | Voice Intonation |
|---|---|---|---|---|---|
| 6) | | | | | |
| A) 9 | A-1) 2nd | A-1) 5 | A-1)  | A-1) Press five (5) | A-1) Average |
| | A-2) 1st | A-2) 1 + 1 + 1 + 1 | A-2)  | A-2) six (1), seven (1), eight (1), nine (1) | A-2) Average |
| B) +6 | B-1) 2nd | B-1) 5 | B-1)  | B-1) Exchange five (10-5) | B-1) Lower |
| | B-2) 1st | B-2) 1 | B-2)  | B-2) six (1) | B-2) Higher |

EXAMPLE: **48 + 8**

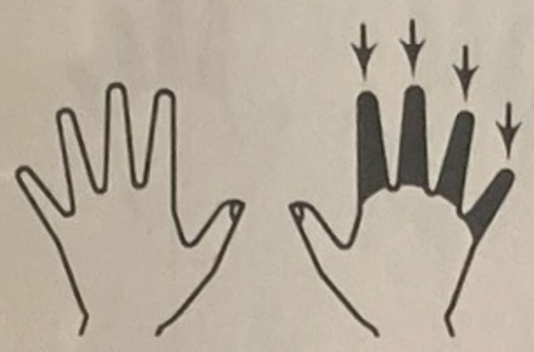
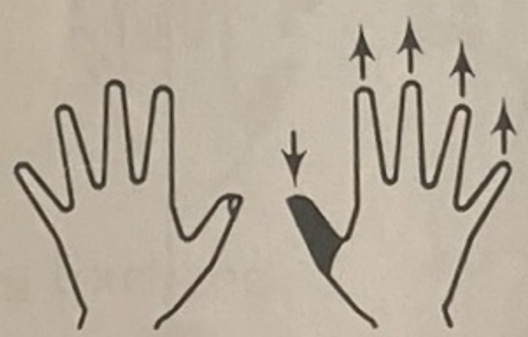
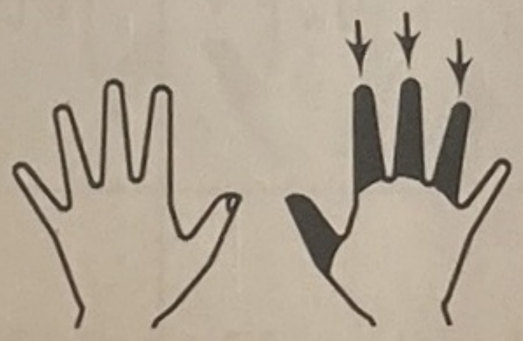
| No. | Breakdown of Addend | Manipulation | Verbal Count |
|---|---|---|---|
| (A) **48** | A-1) 10 + 10 + 10 + 10 | A-1)  | A-1) ten, twenty, thirty, forty |
| | A-2) 5 | A-2)  | A-2) five |
| | A-3) 1 + 1 + 1 | A-3)  | A-3) six, seven, eight. |
| (B) **+8** | B-1) 5 | B-1)  | B-1) Big Exchange 5 |
| | B-2) 1 | B-2)  | B-2) six |

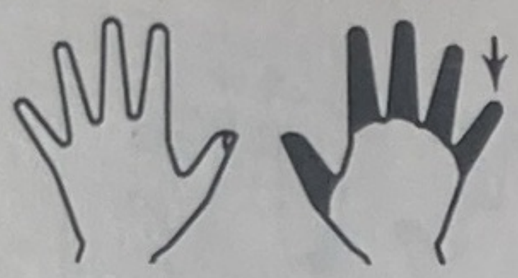
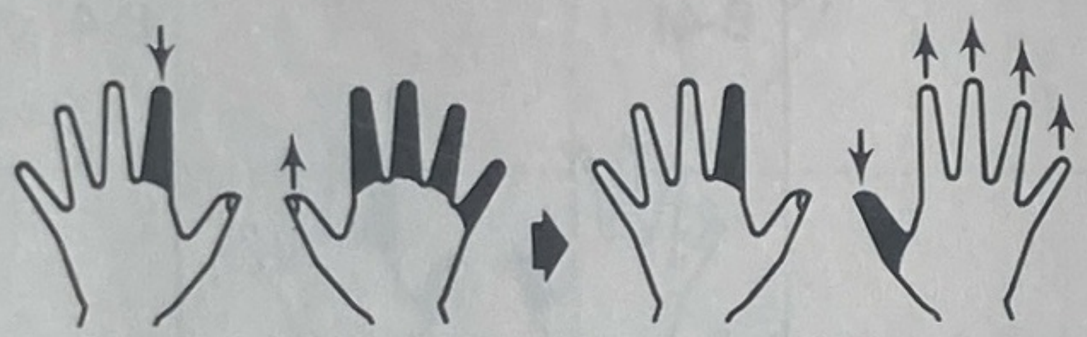
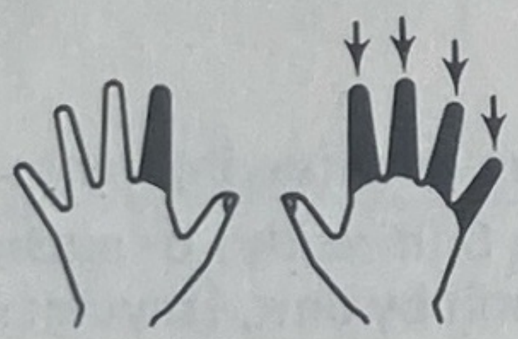
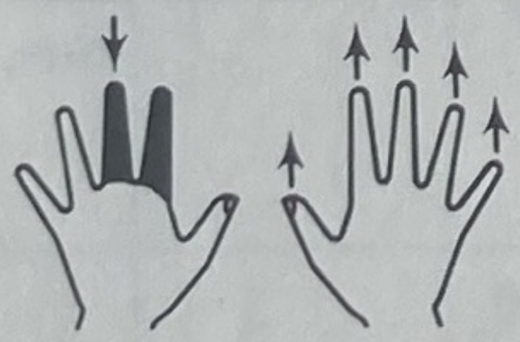
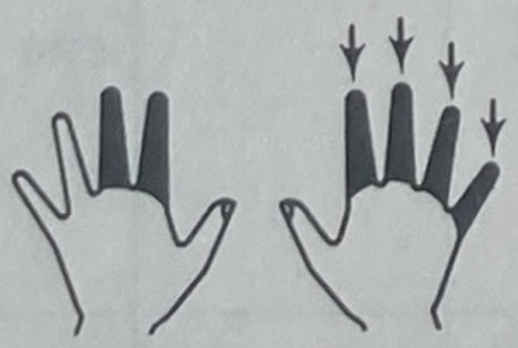
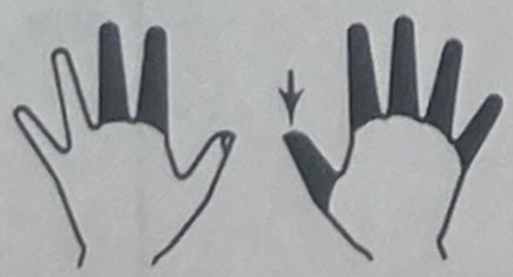
| No. | Breakdown of Addend | Manipulation | Verbal Count |
|---|---|---|---|
| | B-3) 1 | B-3)  | B-3) seven |
| | B-4) 1 | B-4)  | B-4) eight |

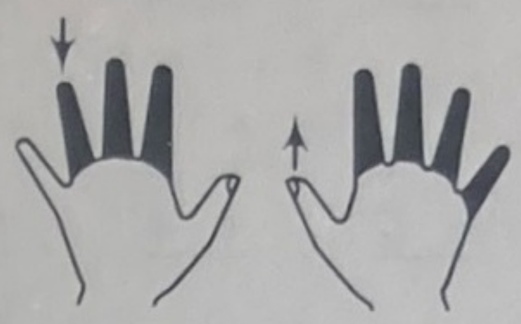
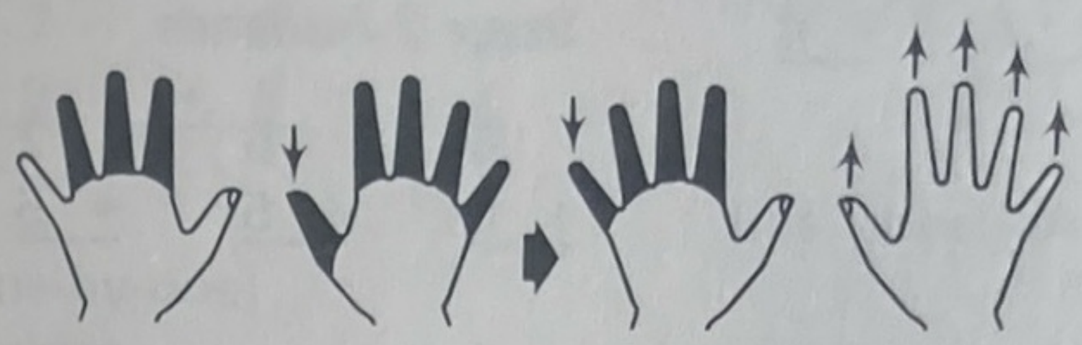
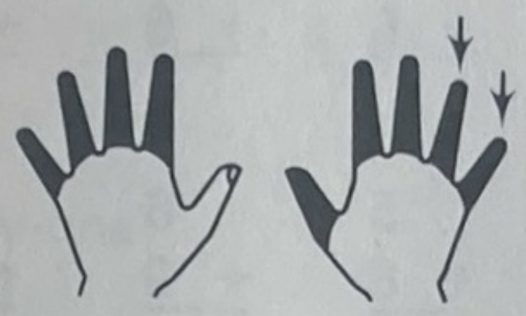
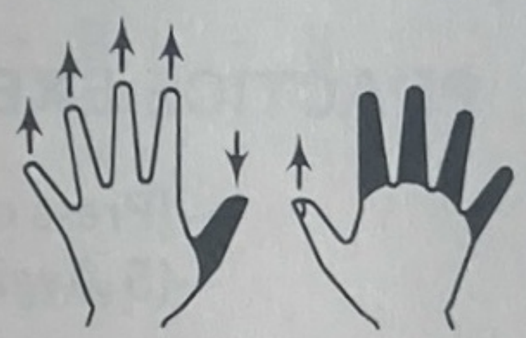
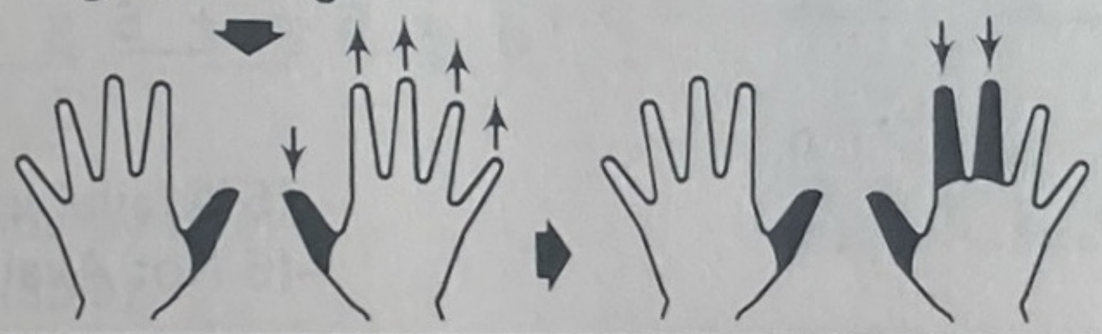
## MANIPULATION EXAMPLES

a) First Stage Numbers—1, 2, 3, 4,—press one-by-one, (saying: one, two, three, four).

b) Second Stage Numbers—5, 6, 7, 8, 9,—press 5 (directly) or exchange 5 (10–5) (saying: five) and then press the rest of the number, unit by unit, [saying: six, seven, eight, nine] .

| Number | Stage | Manipulation | Verbal Count |
|---|---|---|---|
| 4 | 1st |  | one, two, three, four |
| + 1 | 1st |  | one |
| + 3 | 1st |  | one, two, three |

| Number | Stage | Manipulation | Verbal Count |
|---|---|---|---|
| + 1 | 1st |  | one |
| + 6 | 2nd |  | Exchange 5 (10–5), six (1) |
| + 4 | 1st |  | one, two, three, four |
| + 1 | 1st |  | one |
| + 4 | 1st |  | one, two, three, four |
| + 5 | 2nd |  | Press five (5) |

| Number | Stage | Manipulation | Verbal Count |
|---|---|---|---|
| + 5 | 2nd |  | Exchange 5 (10-5) |
| + 6 | 2nd |  | Press five (5), six (1) |
| + 7 | 2nd | | Press five (5), six (1) seven (1) |
| + 2 | 1st |  | one, two |
| + 8 | 2nd |  | *Big Exchange Five (5) six (1), seven (1), eight (1) |
| 57 | | *Big Exchange 5  | |

(Answer)

*Big Exchange Five (5)* Press fifty, clear forty on the left hand; clear five on the right hand simultaneously.

## PRACTICE EXERCISES

Combination of Stage 1 Addends and Stage 1 Addends

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 1 |
| + 1 | + 1 | + 1 | + 1 | + 2 |

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 1 |
| + 3 | + 3 | + 3 | + 3 | + 4 |

Combination of Stage 1 Addends and Stage 2 Addends

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 1 |
| + 5 | + 5 | + 5 | + 5 | + 6 |

| | | | | |
|---|---|---|---|---|
| 2 | 3 | 4 | 1 | 2 |
| + 6 | + 6 | + 6 | + 7 | + 7 |

| | | | | |
|---|---|---|---|---|
| 3 | 4 | 1 | 2 | 3 |
| + 7 | + 7 | + 8 | + 8 | + 8 |

| | | | | |
|---|---|---|---|---|
| 4 | 1 | 2 | 3 | 4 |
| + 8 | + 9 | + 9 | + 9 | + 9 |

Combination of Stage 2 Addends and Stage 1 Addends

| | | | | |
|---|---|---|---|---|
| 5 | 6 | 7 | 8 | 9 |
| + 1 | + 1 | + 1 | + 1 | + 1 |

| | | | | |
|---|---|---|---|---|
| 5 | 6 | 7 | 8 | 9 |
| + 2 | + 2 | + 2 | + 2 | + 2 |

| | | | | |
|---|---|---|---|---|
| 5 | 6 | 7 | 8 | 9 |
| + 3 | + 3 | + 3 | + 3 | + 3 |

| | | | | |
|---|---|---|---|---|
| 5 | 6 | 7 | 8 | 9 |
| + 4 | + 4 | + 4 | + 4 | + 4 |

| | | | | |
|---|---|---|---|---|
| 5 | 6 | 7 | 8 | 9 |
| + 1 | + 2 | + 3 | + 4 | + 4 |

| | | | | |
|---|---|---|---|---|
| 5 | 6 | 7 | 8 | 9 |
| + 2 | + 4 | + 2 | + 1 | + 3 |

Combination of Stage 2 Addends and Stage 2 Addends

| | | | | |
|---|---|---|---|---|
| 5 | 6 | 7 | 8 | 9 |
| + 5 | + 5 | + 5 | + 5 | + 5 |

| | | | | |
|---|---|---|---|---|
| 8 | 9 | 5 | 6 | 7 |
| + 6 | + 6 | + 7 | + 7 | + 7 |

| | | | | |
|---|---|---|---|---|
| 6 | 7 | 8 | 9 | 5 |
| + 8 | + 8 | + 8 | + 8 | + 9 |

| | | | | |
|---|---|---|---|---|
| 5 | 6 | 7 | 8 | 9 |
| + 6 | + 6 | + 6 | + 7 | + 7 |

| | | | | |
|---|---|---|---|---|
| 5 | 6 | 7 | 8 | 9 |
| + 8 | + 9 | + 9 | + 9 | + 9 |

## PRACTICE EXERCISES

—(Press one-by-one)
—(5 Available)

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| + 5 | + 5 | + 5 | + 5 |

—(5 Available)
—(5 Not Available)

| | | | | |
|---|---|---|---|---|
| 5 | 6 | 7 | 8 | 9 |
| + 5 | + 5 | + 5 | + 5 | + 5 |

—(Press one-by-one)
—(5 Available)

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| + 6 | + 6 | + 6 | + 6 |

—(5 Available)
—(5 Not Available)

| | | | | |
|---|---|---|---|---|
| 5 | 6 | 7 | 8 | 9 |
| + 6 | + 6 | + 6 | + 6 | + 6 |

—(Press one-by-one)
—(5 Available)

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| + 7 | + 7 | + 7 | + 7 |

—(5 Available)
—(5 Not Available)

| | | | | |
|---|---|---|---|---|
| 5 | 6 | 7 | 8 | 9 |
| + 7 | + 7 | + 7 | + 7 | + 7 |

—(Press one-by-one)
—(5 Available)

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| + 8 | + 8 | + 8 | + 8 |

—(5 Available)
—(5 Not Available)

| | | | | |
|---|---|---|---|---|
| 5 | 6 | 7 | 8 | 9 |
| + 8 | + 8 | + 8 | + 8 | + 8 |

—(Press one-by-one)
—(5 Available)

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| + 9 | + 9 | + 9 | + 9 |

—(5 Available)
—(5 Not Available)

| | | | | |
|---|---|---|---|---|
| 5 | 6 | 7 | 8 | 9 |
| + 9 | + 9 | + 9 | + 9 | + 9 |

## PRACTICE EXERCISES

When number 5 is marked "• ," press 5th finger (5) saying press five.

| | | | |
|---|---|---|---|
| 1 | 2 • 5 | 2 | |
| 1 | 1 | 1 • 5 | |
| + • 5 | + • 5 | + 2 | + 2 |

| | | | |
|---|---|---|---|
| 4 • 5 | | 3 • 5 | |
| • 5 | 4 | 1 | 3 |
| + 1 | + 3 | + • 5 | + 2 |

| | | | |
|---|---|---|---|
| 2 | 4 • 5 | • 5 | |
| • 5 | • 5 | 1 | 1 |
| + 4 | + 4 | + 3 | + 4 |

| | | | |
|---|---|---|---|
| • 5 | 4 | 1 | 3 |
| 2 • 5 | | 3 • 5 | |
| + 3 | + 2 | + • 5 | + 2 |

When number 5 is marked " ◉ ," exchange five (10–5) saying exchange five.

| | | | |
|---|---|---|---|
| 5 | 5 | 1 | 4 |
| ◉ 5 | ◉ 5 | 4 | 3 |
| + 5 | + 4 | + ◉ 5 | + ◉ 5 |

| | | | |
|---|---|---|---|
| 5 | 5 | 3 | 4 |
| ◉ 5 | ◉ 5 | 2 | 4 |
| + 3 | + 2 | + ◉ 5 | + ◉ 5 |

| | | | |
|---|---|---|---|
| 2 | 1 | 4 | 4 |
| 5 | 5 | 2 | 1 |
| + ◉ 5 | + ◉ 5 | + ◉ 5 | + ◉ 5 |

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 4 | 4 | 5 |
| + ⊙ 5 | + ⊙ 5 | + ⊙ 5 | + ⊙ 5 |

| | | | |
|---|---|---|---|
| 5 | 1 | 6 | 3 |
| 4 | 4 | 3 | 5 |
| + ⊙ 6 | + ⊙ 6 | + ⊙ 6 | + ⊙ 6 |

When number 6 is marked "• ," press 5th finger and 1 more saying press five—six.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 1 |
| 2 | 1 | 1 | 3 |
| + • 6 | + • 6 | + • 6 | + • 6 |

| | | | |
|---|---|---|---|
| 4 | 4 | 4 | 4 |
| • 6 | • 6 | • 6 | • 6 |
| + • 6 | + 4 | + 3 | + 2 |

| | | | |
|---|---|---|---|
| 4 | 2 | 1 | 3 |
| • 6 | 2 | 1 | • 6 |
| + 5 | + • 6 | + • 6 | + 5 |

| | | | |
|---|---|---|---|
| 2 | 1 | 5 | 2 |
| • 6 | • 6 | 5 | • 6 |
| + 5 | + 5 | + • 6 | + 1 |

When number 7 is marked "• ," press 5th finger and 2 more unit by unit saying press five—six—seven.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| • 7 | • 7 | • 7 | • 7 |
| + 4 | + 5 | + • 7 | + • 7 |

| | | | |
|---|---|---|---|
| 1 | 2 | 5 | • 7 |
| 3 | 2 | 5 | 5 |
| + • 7 | + • 7 | + • 7 | + • 7 |

| | | | |
|---|---|---|---|
| 3 | 4 | 3 | 2 |
| 1 | • 7 | • 7 | • 7 |
| + • 7 | + 1 | + 5 | + 6 |

| | | | |
|---|---|---|---|
| 6 | 4 | • 7 | 2 |
| 5 | • 7 | 5 | • 7 |
| + • 7 | + 6 | + 6 | + 1 |

When number 6 is marked "⊙ ," exchange five (10–5) and then press 1 more saying exchange five—six.

| | | | |
|---|---|---|---|
| 5 | 6 | 5 | 3 |
| ⊙ 6 | ⊙ 6 | ⊙ 6 | 6 |
| + 6 | + 6 | + 5 | + ⊙ 6 |

| | | | |
|---|---|---|---|
| 1 | 4 | 2 | 2 |
| 6 | 1 | 5 | 6 |
| + ⊙ 6 | + ⊙ 6 | + ⊙ 6 | + ⊙ 6 |

| | | | |
|---|---|---|---|
| 4 | 3 | 4 | 2 |
| 2 | 3 | 5 | 3 |
| + ⊙ 6 | + ⊙ 6 | + ⊙ 6 | + ⊙ 6 |

When number 7 is marked "⊙ ," exchange five (10–5) and 2 more unit by unit saying exchange five—six—seven.

| | | | |
|---|---|---|---|
| 1 | 7 | 7 | 7 |
| 4 | ⊙ 7 | ⊙ 7 | ⊙ 7 |
| + ⊙ 7 | + 7 | + 3 | + 1 |

| | | | |
|---|---|---|---|
| 7 | 1 | 7 | 6 |
| ⊙ 7 | 7 | 1 | ⊙ 7 |
| + 5 | + ⊙ 7 | + ⊙ 7 | + 5 |

| | | | |
|---|---|---|---|
| 3 | 2 | 5 | 7 |
| 2 | 3 | 3 | ⊙ 7 |
| + ⊙ 7 | + ⊙ 7 | + ⊙ 7 | + 2 |

| | | | |
|---|---|---|---|
| 6 | 4 | 2 | 7 |
| 3 | 5 | 7 | 1 |
| + • 7 | + • 7 | + • 7 | + • 7 |

When number 8 is marked "• ," press 5th finger (5) and then press 3 more unit by unit saying press five—six—seven—eight.

| | | | |
|---|---|---|---|
| 4 | • 8 | 5 | 7 |
| • 8 | 1 | 7 | 6 |
| + 1 | + 5 | + • 8 | + • 8 |

| | | | |
|---|---|---|---|
| 4 | 4 | • 8 | • 8 |
| • 8 | • 8 | 2 | 3 |
| + 2 | + • 8 | + • 8 | + • 8 |

| | | | |
|---|---|---|---|
| • 8 | 6 | 2 | 3 |
| 3 | 7 | 2 | • 8 |
| + 4 | + • 8 | + • 8 | + 1 |

| | | | |
|---|---|---|---|
| • 8 | 7 | 5 | • 8 |
| 6 | 7 | 6 | 5 |
| + • 8 | + • 8 | + • 8 | + • 8 |

When number 8 is marked "⊙ ," exchange five (10-5) and then press 3 more unit by unit saying exchange five—six—seven—eight.

| | | | |
|---|---|---|---|
| 2 | 3 | 1 | 5 |
| 7 | 6 | 8 | 4 |
| + ⊙ 8 | + ⊙ 8 | + ⊙ 8 | + ⊙ 8 |

| | | | |
|---|---|---|---|
| 1 | 5 | 8 | 6 |
| 4 | ⊙ 8 | 7 | ⊙ 8 |
| + ⊙ 8 | + 8 | + ⊙ 8 | + 8 |

| | | | |
|---|---|---|---|
| 4 | 6 | 7 | 5 |
| 2 | ⊙ 8 | 1 | ⊙ 8 |
| + ⊙ 8 | + 1 | + ⊙ 8 | + 5 |

| | | | |
|---|---|---|---|
| 6 | 7 | 8 | 8 |
| • 8 | • 8 | • 8 | 1 |
| + 4 | + • 8 | + • 8 | + • 8 |

When number 9 is marked "• ," press 5th finger (5) and then press 4 more unit by unit saying press five—six—seven—eight—nine.

| | | | |
|---|---|---|---|
| 7 | 8 | 5 | 4 |
| 5 | 6 | 7 | 8 |
| + • 9 | + • 9 | + • 9 | + • 9 |

| | | | |
|---|---|---|---|
| 5 | 6 | 3 | 2 |
| 8 | 8 | 1 | 2 |
| + • 9 | + • 9 | + • 9 | + • 9 |

| | | | |
|---|---|---|---|
| • 9 | • 9 | 2 | • 9 |
| 4 | 1 | • 9 | 3 |
| + • 9 | + • 9 | + • 9 | + • 9 |

| | | | |
|---|---|---|---|
| 3 | 4 | 1 | 5 |
| • 9 | • 9 | • 9 | 6 |
| + • 9 | + • 9 | + 5 | + • 9 |

When number 9 is marked "⊙ ," exchange five (10-5) and then press 4 more unit by unit saying exchange five—six—seven—eight—nine.

| | | | |
|---|---|---|---|
| 5 | 6 | 7 | 8 |
| ⊙ 9 | ⊙ 9 | ⊙ 9 | ⊙ 9 |
| + 9 | + ⊙ 9 | + ⊙ 9 | + ⊙ 9 |

| | | | |
|---|---|---|---|
| 9 | 5 | 1 | 4 |
| ⊙ 9 | 1 | 4 | 4 |
| + ⊙ 9 | + ⊙ 9 | + ⊙ 9 | + ⊙ 9 |

| | | | |
|---|---|---|---|
| 2 | 4 | 6 | 7 |
| 3 | 1 | ⊙ 9 | 8 |
| + ⊙ 9 | + • 9 | + 4 | + ⊙ 9 |

| | | | | |
|---|---|---|---|---|
| | 7 | 4 | 6 | 1 |
| | ⊙ 9 | 5 | 3 | 7 |
| + | 5 | + ⊙ 9 | + ⊙ 9 | + ⊙ 9 |

## Hand Transfer

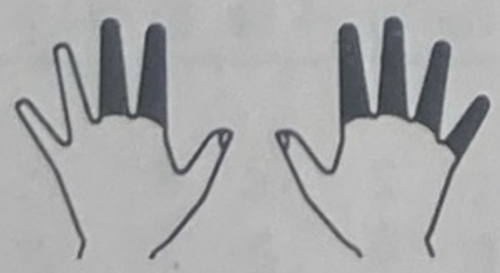
To increase speed the carry-over box is not used in the second stage. Instead, we use the hand transfer. The right hand reveals the answer for the column that is being added, and after that answer has been written down the right hand is free. The left-hand carry-over number is then transferred to the right hand. Different methods are employed for right-handed and left-handed children.

*Right-Handed Children*—The child reads the right hand only after totaling a column and then writes down the answer pressed on the right hand. Before going on to add the next column, the child transfers the carry-over number from the left to the right hand, which now becomes the next place column. Once the number is transferred, be sure to clear the left hand.

*Left-Handed Children*—The child calls out the number on the right hand and almost simultaneously transfers the carry-over number from the left hand to the right hand as he writes down the right-hand number with his left hand.

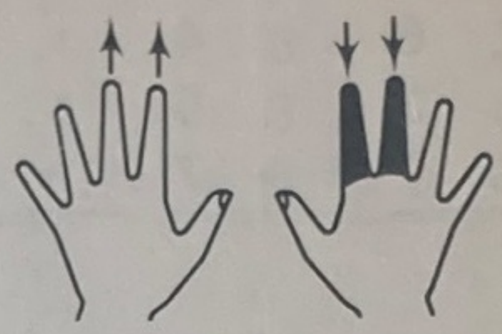
**Note:** In the last part of 2nd stage, it is also recommended that the child hold his pencil while calculating to increase his speed.

(A) 247
569
186
932



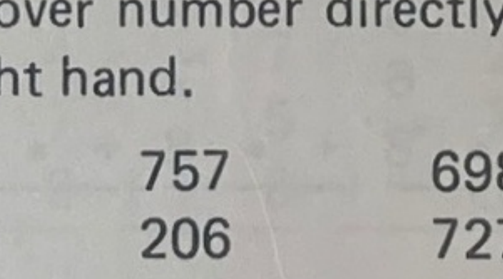
Ones Calculated

(B-1) 247
569
186
932
4



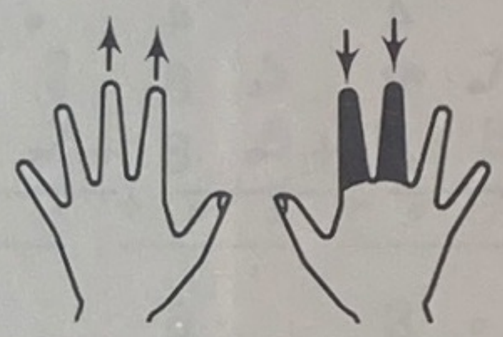
Ready to Calculate Tens

(B-2) 247
569
186
932
4



Tens Calculated

(C) 247
569
186
932
34



Ready to Calculate Hundreds

## PRACTICE EXERCISES

Transfer carry-over number directly from left hand to right hand.

| | | |
|---|---|---|
| 943 | 757 | 698 |
| 720 | 206 | 727 |
| 672 | 713 | 865 |
| + 586 | + 909 | + 521 |
| 337 | 675 | 708 |
| 456 | 200 | 123 |
| 912 | 912 | 497 |
| + 885 | + 345 | + 185 |
| 278 | 936 | 526 |
| 695 | 387 | 810 |
| 273 | 165 | 647 |
| + 804 | + 794 | + 338 |
| 159 | 218 | 432 |
| 436 | 643 | 156 |
| 207 | 579 | 907 |
| + 895 | + 600 | + 583 |

Transfer carry-over number directly from left hand to right hand.

| | | | |
|---|---|---|---|
| 240 | 578 | 912 | 315 |
| 693 | 943 | 426 | 624 |
| 187 | 185 | 343 | 709 |
| + 526 | + 206 | + 557 | + 458 |

| | | | |
|---|---|---|---|
| 968 | 100 | 726 | 876 |
| 703 | 674 | 894 | 942 |
| 215 | 896 | 360 | 108 |
| + 492 | + 532 | + 514 | + 355 |

| | | | |
|---|---|---|---|
| 694 | 321 | 543 | 959 |
| 201 | 756 | 279 | 201 |
| 573 | 890 | 648 | 428 |
| + 687 | + 543 | + 106 | + 637 |

| | | | |
|---|---|---|---|
| 182 | 986 | 760 | 456 |
| 496 | 624 | 203 | 723 |
| 535 | 308 | 185 | 698 |
| + 708 | + 157 | + 943 | + 501 |

## RETRIEVAL OF TRANSFERRED CARRY-OVER NUMBERS

Occasionally, when performing problems with many addends, a mistake is made after most of the problem is completed. Suppose when calculating the problem in the illustration, you arrive at the answer through the hundreds' column, but then get distracted while computing the thousands' column. Since you have been directly transferring the carry-over numbers, you may have forgotten what number was carried over from the hundreds' to the thousands' column. However, you do not have to go back and completely recalculate from the units' column. Here is a short cut.

Lost carry-over into the thousands' column

| | | | | |
|---|---|---|---|---|
| | □ | □ | ← | Lost carry-over into hundreds' column |
| | 7, | 6 | 9 | 4 | |
| | 1, | 9 | 4 | 5 | |
| | 3, | 8 | 5 | 8 | |
| + | 8, | 6 | 7 | 2 | |
| | 1 | 6 | 9 | | |

Draw in carry-over boxes over the hundreds' and thousands' columns. Adding the hundreds' column, you find that you reach a total of 29 on your fingers. However, the answer below the line is a 1. You should therefore treat the carry-over box in the hundreds' column as a missing addend. In order for your right hand to register a 1, you must add 2 more to the hundreds' column. The 2 you have added is the carry-over missing in the hundreds' column. Since your left hand is now registering 30, you know that a 3 is carried over to the thousands' column. You have found the number you needed, "3."

You will note that the carry-over number in the thousands' column was the one that you were searching for. In the process, you also found the carry-over number in the hundreds' column. While you didn't care what was the missing carry-over number in the hundreds' column, you had to find it in order to make sure that you were accurately finding the carry-over number in the thousands' column.

## MULTIPLICATION STAGE 2

Multiplication Stage 2 is merely a statement of the multiplication tables. There are, however, some variations from the conventional presentation.

The main difference occurs because of the emphasis Chisanbop ™ places on the decimal system. Although conventional math teaches the decimal system, vestiges

of an earlier counting system based on twelve still survive in our grade school text books. Most of these texts still have tables for the numbers ten, eleven, and twelve. Note that all three of these numbers are two digits and can no longer be called units. Stopping at twelve is obviously arbitrary. Why not a table for thirteen? Chisanbop ™ tables stop at nine for several reasons. In the integer number system, the largest number in any column is 9. To go higher is to reach a carry-over number and be forced to the next left-hand column.

When we multiply any two-digit number by a unit, we usually do it column by column. For example, when multiplying 27 x 9, we first multiply 7 x 9 and then 2 x 9. We also take account of place value by placing the result of the second multiplication, one column to the left of the first result. The only exceptions to this rule are 10 x 8, 11 x 8, 12 x 8, and possibly 20 x 8, 30 x 8...90 x 8.

Allowing this exception destroys consistency, adds another rule for the child to learn, and makes both rules harder since they must always make an examination to see which rule applies. Adults can perform these exceptions since they were taught 12 tables. If they had been taught 13 tables, they could multiply 13 x 8 directly. Rather than teach a small group of exceptions, Chisanbop ™ teaches the tables up to 9 and then gives one consistent rule for multi-digit multiplication. The consistent rule will be explained in Multiplication Stages 3 and 4. For now, we explain the Chisanbop ™ tables.

In Stage 2 the tables should be taught in the following order: 0, 1, 5, 9, 6, 4, 2*, 8, 3, 7.

*The table for 2's can be taught early in Multiplication Stage 1 using the following steps.

1. Press, unit by unit, (one finger at a time) up to twenty saying: *"one two, one two, one two,...one two."*
2. Press twos in pairs, simultaneously, up to twenty saying: *"one two, one two, one two,...one two."* (Initially when going from 4 to 6, 8 to 10, 14 to 16, and 18 to 20, the children will press unit by unit; later they will press these manipulations simultaneously.
3. Press twos in pairs, simultaneously, up to twenty saying: *"two, two, two,...two."*
4. Press two in pairs, simultaneously up to twenty, saying: *"one* (2), *two* (2's), *three* (2's),...*ten* (2's)." Care should be taken that the student is comfortable at each step before moving on to the next one.
5. Press two in pairs, simultaneously up to twenty, saying: *"one, two, three,...ten."*

## METHOD OF TEACHING THE TABLES

In the classroom, the tables must be taught in steps. For the purpose of illustration, we will use the six unit table.

*Step 1.* Press sixes to sixty, saying: *"press five and press one more, six; exchange five and press one more, six; press five and one more, six; exchange five and one more, six; press ten and clear four, six; press five and one more, six; exchange five and one more, six; press five and one more, six; make a Big Exchange Five and one more, six; press ten*

*and clear four, six.* Note the instructions when going from 24 to 30 and 54 to 60. On both occasions, the manipulations can be broken down to press five (with the fingers now reading 29 or 59) and one more, bringing the finger totals to 30 or 60. However pressing 10 and clearing 4 is the easier and quicker manipulation. It is the easiest to remember. You should therefore encourage the students to simultaneously press 10 and clear 4. It is often advisable to stop the pressing at 24, 30 and 54 and ask the class to read their fingers since difficult manipulations are being performed. The table is self-checking. After pressing ten sixes, the students should all have sixty registered on their fingers. This should be practiced until the students are completely secure in the manipulations.

*Step 2.* Once the students are secure in the tables, they should be timed to make sure they add ten sixes in fifteen seconds.

*Step 3.* The children themselves, will lead the way in moving on to step 3. They will begin to press six simultaneously, instead of pressing or exchanging a five and then pressing one more.

*Step 4.* Once the students reach the point where they consistently press six simultaneously, the oral instructions should be changed to "*one* (six), *two* (sixes), *three* (sixes)...*ten* (sixes)."

*Step 5.* PRESS SIXES SIMULTANEOUSLY. In the oral count delete the expression of the unit (here "six"). Thus say "one, two, three...ten."

Throughout these steps, a teacher should feel free to stop at any moment and say "Read your fingers." In steps one and two, it is to make sure that the students have performed the difficult manipulations; in steps four and five it is for the purpose of beginning to internalize the tables.

Thus the teacher in step 4 might say: "Press sixes to sixty. Ready, one (six), two (sixes), three (sixes). Read your fingers. What are three sixes?" (Response: "18"). "Good. Four (sixes), five (sixes)... ten (sixes). Read your fingers." (Response: "60"). "Good."

It should be noted that these Chisanbop ™ tables offer a bright alternative to the pure rote system of memorizing the tables. The rhythmic pattern of the tables are actually an inducement and experience shows very early internalization.

## Teaching Sequence for the Tables

Some tables are easier to learn than others. Therefore, there is a definite sequence in which they should be taught. As mentioned earlier, the tables should be taught in the following order 0, 1, 5, 9, 6, 4, 2, 8, 3, and 7.

Only one table should be taught at a time. The nines should not be introduced until after the fives are known thoroughly. The first thing that must be learned is the finger manipulations. Teachers should not be concerned with the memorization of the multiplication facts when a table is first introduced. They should care that the students are able to understand and make each progressive manipulation. Only after a student is able to press a table within 15 seconds while counting how many times he has added the unit (Time Progress Chart (B)), should the student begin to concentrate on the number facts. The memorization will come

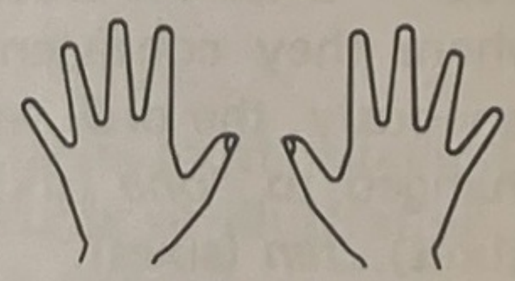
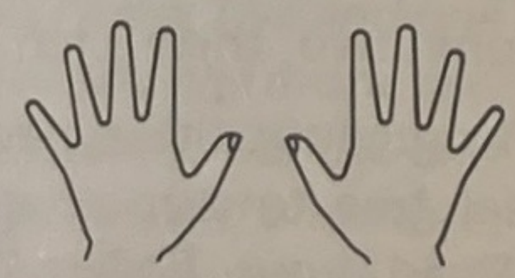
very rapidly. Once the students can press a table in 15 seconds while counting, they should be given dozens of problems using their new table exclusively. This will reinforce their physical memory as they continue to press the table and also begin to achieve internalization. They can also be given problems of a multi-digit number times a single digit number utilizing the HYPAI Multiplication Format as seen on the following pages. Once again, however, the single digit should be a number from the table you are concentrating on.

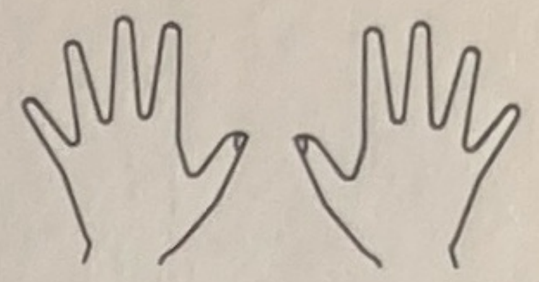
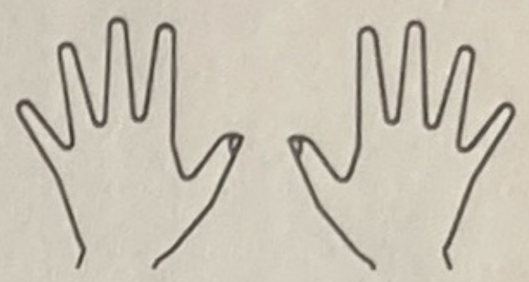
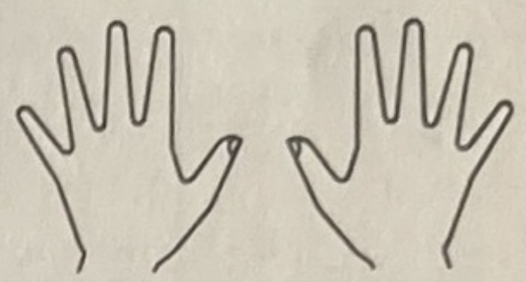
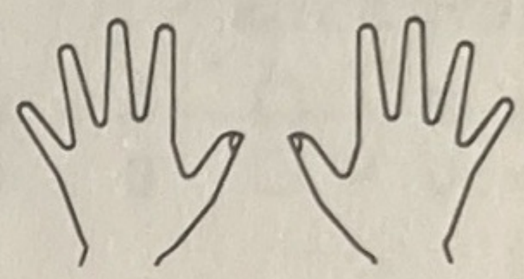
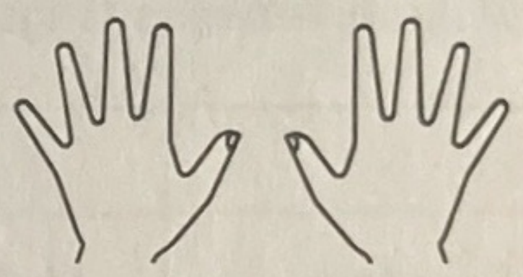
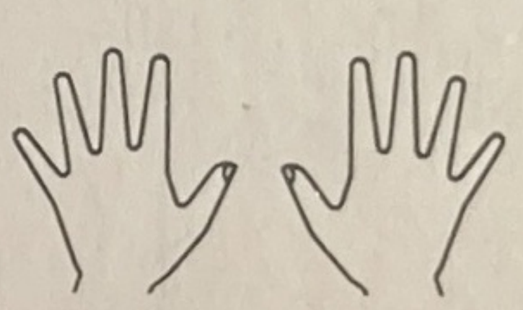
Eventually, the students must be introduced to problems where the tables are mixed. Here we have introduced this concept after each pair of tables. When you first give students pages which combine pairs of tables, have them practice both tables before starting the problems. If they can not work out the problems, it means you are teaching the tables too rapidly. Go back and concentrate on just one table. Eventually, you will be able to introduce them to problems that require three tables, and then to problems requiring four tables.

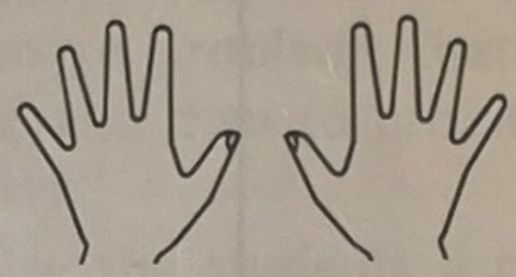
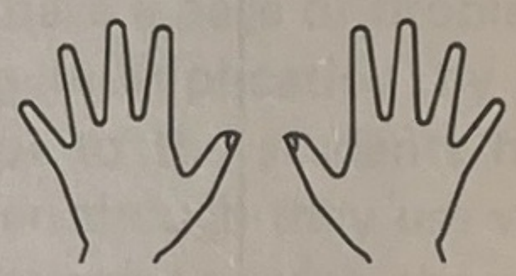
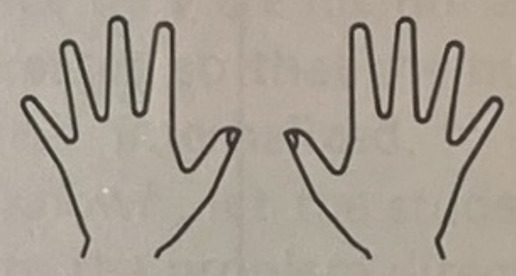
When you first give the students a page with two tables, take advantage of the problems to teach the commutative rule. Thus if you have a page of problems that are featuring multiplication by 6's and 4's, point out to the students how 6 x 4 = 4 x 6. Even though they use very different manipulations to reach the product, the product still remains the same.

Do not make the students continue to press the tables once they are memorized. The tables were created so that the number facts could be internalized. If the number facts are known, let the students write the answers to the problem directly. You will be surprised how quickly they will be able to do this.

## ZEROES

| Problem | Unit | Pupil Counts How Many Times | Manipulation | Fingers Read |
|---|---|---|---|---|
| **0 x 0** | – | "Zero" (Zero) | No Manipulation |  **0** |
| **1 x 0** | 0 | "One" (Zero) | No Manipulation |  **0** |

| | | | | |
|---|---|---|---|---|
| **2 x 0** | +0 | "Two" (Zeroes) | No Manipulation |  **0** |
| **3 x 0** | +0 | "Three" (Zeroes) | No Manipulation |  **0** |
| **4 x 0** | +0 | "Four" (Zeroes) | No Manipulation |  **0** |
| **5 x 0** | +0 | "Five" (Zeroes) | No Manipulation |  **0** |
| **6 x 0** | +0 | "Six" (Zeroes) | No Manipulation |  **0** |
| **7 x 0** | +0 | "Seven" (Zeroes) | No Manipulation |  **0** |

| | | | | |
|---|---|---|---|---|
| **8 x 0** | +0 | "Eight" (Zeroes) | No Manipulation |  **0** |
| **9 x 0** | +0 | "Nine" (Zeroes) | No Manipulation |  **0** |
| **10 x 0** | +0 | "Ten" (Zeroes) | No Manipulation |  **0** |

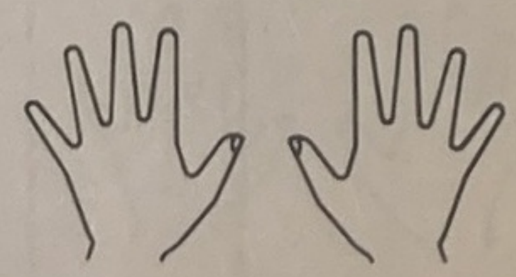
Total: 0

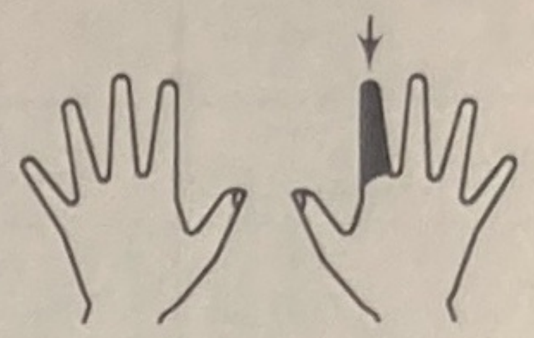
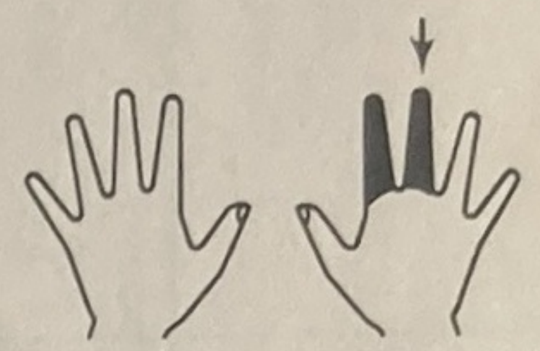
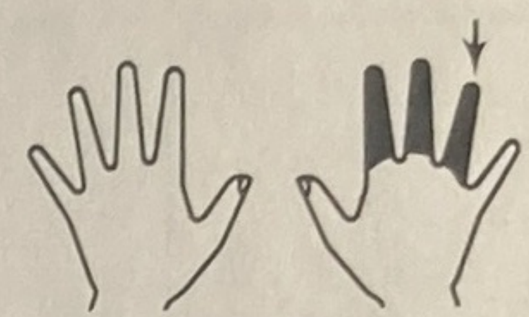
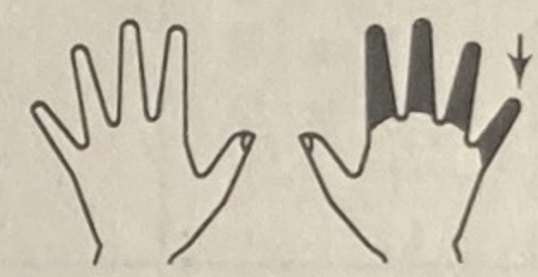
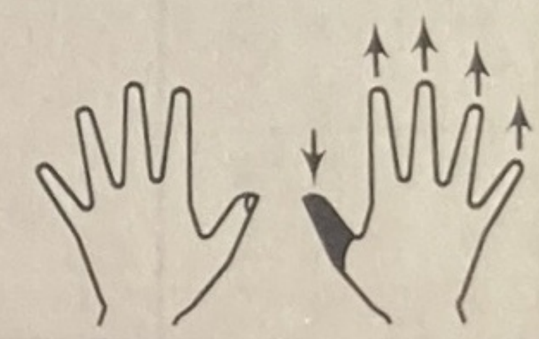
## EXERCISES

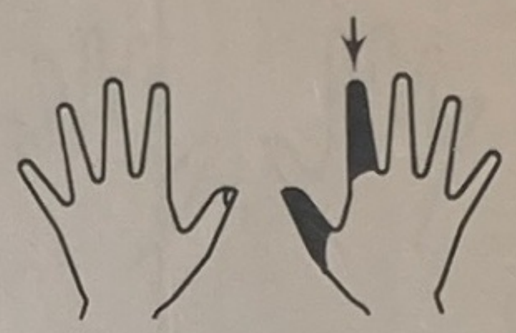
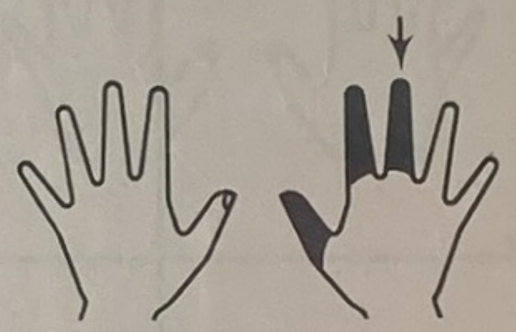
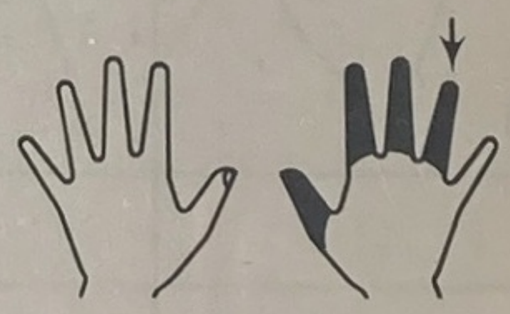
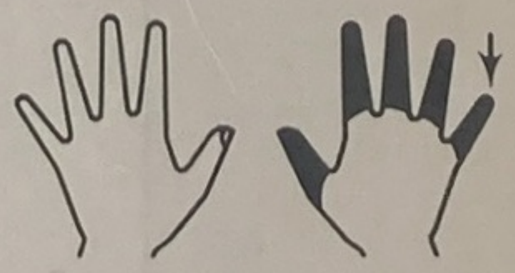
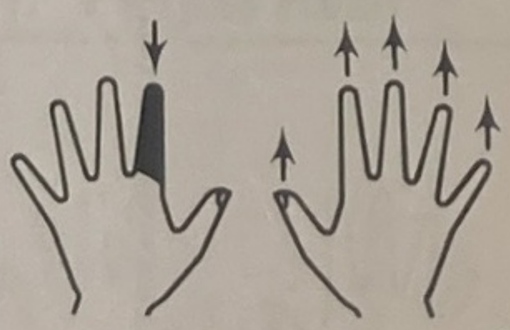
| | | | | |
|---|---|---|---|---|
| 0 x 0 = ☐ | 4 x 0 = ☐ | 8 x 0 = ☐ | 3 x 0 = ☐ | 7 x 0 = ☐ |
| 2 x 0 = ☐ | 6 x 0 = ☐ | 1 x 0 = ☐ | 5 x 0 = ☐ | 9 x 0 = ☐ |

## ONES

All manipulations should be made simultaneously.
*Available—means fingers are available for Pressing.

| Problem | Unit | Pupil Counts How Many Times | Manipulation | Fingers Read |
|---|---|---|---|---|
| **0 x 1** | – | "Zero" (One) | No Manipulation |  **0** |

| | | | | |
|---|---|---|---|---|
| **1 x 1** | 1 | "One"<br>(One) | Press 1<br>*(Available) |  <br>1 |
| **2 x 1** | +1 | "Two"<br>(Ones) | Press 1<br>*(Available) |  <br>2 |
| **3 x 1** | +1 | "Three"<br>(Ones) | Press 1<br>*(Available) |  <br>3 |
| **4 x 1** | +1 | "Four"<br>(Ones) | Press 1<br>*(Available) |  <br>4 |
| **5 x 1** | +1 | "Five"<br>(Ones) | Press 5 & Clear 4<br>(5-4) |  <br>5 |

| | | | | |
|---|---|---|---|---|
| **6 x 1** | +1 | "Six" (Ones) | Press 1 *(Available) |  6 |
| **7 x 1** | +1 | "Seven" (Ones) | Press 1 *(Available) |  7 |
| **8 x 1** | +1 | "Eight" (Ones) | Press 1 *(Available) |  8 |
| **9 x 1** | +1 | "Nine" (Ones) | Press 1 *(Available) |  9 |
| **10 x 1** | +1 | "Ten" (Ones) | **Exchange 5 & Clear 4 (A) 10-5 (B) -4 |  10 |
| | Total: 10 | | | |

(A) MULTIPLICATION STAGE **2**–Time Progress Chart (A)

Pressing 10 Times (*Without* Counting As The Unit Is Pressed)

| Unit \ Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **1** | **10** | (time) | | | | | | | | | |

**NOTE:** You can proceed only after you have done the time progress chart (A) within 15 seconds per trial.

(B) MULTIPLICATION STAGE 2–Time Progress Chart (B)

(*While* Counting How Many Times The Unit Is Pressed)

| Unit \ Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **1** | **10** | (time) | | | | | | | | | |

## EXERCISES

| | | | |
|---|---|---|---|
| 0 x 1 = ☐ | 1 x 1 = ☐ | 4 x 1 = ☐ | 8 x 0 = ☐ |
| 2 x 1 = ☐ | 3 x 1 = ☐ | 2 x 0 = ☐ | 0 x 1 = ☐ |
| 4 x 1 = ☐ | 5 x 1 = ☐ | 9 x 1 = ☐ | 7 x 0 = ☐ |
| 6 x 1 = ☐ | 7 x 1 = ☐ | 5 x 0 = ☐ | 1 x 1 = ☐ |
| 8 x 1 = ☐ | 9 x 1 = ☐ | 3 x 1 = ☐ | 6 x 0 = ☐ |
| 6 x 1 = ☐ | 5 x 0 = ☐ | 9 x 1 = ☐ | 3 x 0 = ☐ |
| 8 x 0 = ☐ | 7 x 1 = ☐ | 2 x 0 = ☐ | 8 x 1 = ☐ |
| 3 x 1 = ☐ | 2 x 0 = ☐ | 0 x 1 = ☐ | 4 x 0 = ☐ |
| 1 x 0 = ☐ | 0 x 1 = ☐ | 1 x 0 = ☐ | 5 x 1 = ☐ |
| 4 x 1 = ☐ | 9 x 0 = ☐ | 6 x 1 = ☐ | 7 x 0 = ☐ |
| 3 x 1 = ☐ | 6 x 0 = ☐ | 4 x 1 = ☐ | 0 x 0 = ☐ |
| 2 x 0 = ☐ | 1 x 1 = ☐ | 8 x 0 = ☐ | 3 x 1 = ☐ |
| 7 x 1 = ☐ | 0 x 0 = ☐ | 1 x 1 = ☐ | 5 x 0 = ☐ |
| 8 x 0 = ☐ | 9 x 1 = ☐ | 7 x 0 = ☐ | 6 x 1 = ☐ |
| 5 x 1 = ☐ | 4 x 0 = ☐ | 9 x 1 = ☐ | 2 x 0 = ☐ |
| 7 x 1 = ☐ | 5 x 0 = ☐ | 1 x 1 = ☐ | 9 x 0 = ☐ |
| 3 x 0 = ☐ | 9 x 1 = ☐ | 6 x 0 = ☐ | 2 x 1 = ☐ |
| 1 x 1 = ☐ | 2 x 0 = ☐ | 4 x 1 = ☐ | 7 x 0 = ☐ |
| 8 x 0 = ☐ | 6 x 1 = ☐ | 5 x 0 = ☐ | 0 x 1 = ☐ |
| 0 x 1 = ☐ | 4 x 0 = ☐ | 3 x 1 = ☐ | 8 x 0 = ☐ |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 | x | 1 | = ☐ | 3 | x | 0 | = ☐ |
| 4 | x | 0 | = ☐ | 5 | x | 1 | = ☐ |
| 9 | x | 1 | = ☐ | 7 | x | 0 | = ☐ |
| 2 | x | 0 | = ☐ | 1 | x | 1 | = ☐ |
| 0 | x | 1 | = ☐ | 8 | x | 0 | = ☐ |

## MULTI-DIGIT MULTIPLICATION —HYPAI FORMAT

As soon as the students have learned the physical manipulations for a table, they can begin to multiply multi-digit numbers with that single digit. They find multiplying the larger numbers challenging and a nice change from the single digit problems. However, since the students are only just beginning to memorize their multiplication tables, special care must be taken in designing a format for them.

Once again, it is helpful to show how multiplication is repetitive addition. Thus the problem 317 x 5 could be visualized as

```
☐☐☐      Carry-over boxes
  3 1 7
  3 1 7
  3 1 7
  3 1 7
+ 3 1 7
-------
  ☐☐☐    Column Answers
```

From the first illustration, it is easy to see how the multiplication problem can be rewritten as

```
☐  ☐    ☐        Carry-over
                    Boxes
   300   10    7
 x   5 x   5 x   5
-------------------
   ☐     ☐     ☐   Column
                    Answers
```

It is quite apparent how the HYPAI Multi-digit Format was developed as seen in the final example.

```
    3 1 7
  x     5
-----------
  ☐☐☐       Carry-over Boxes
+  ☐☐☐      Column Answers
-----------
```

Sum

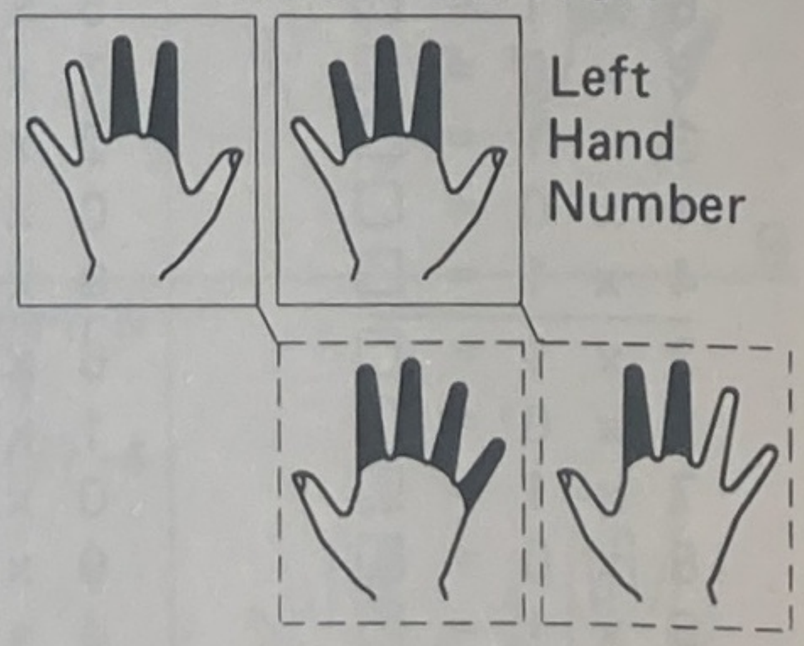
The HYPAI Format was designed with several points in mind. The first point was that as students are only just beginning to learn their multiplicative number facts (indeed they may have few of them known and can only tell them by pressing the tables) they have trouble remembering the carry-over number and then multiplying. The second point was that sometimes when the carry-over number was written above the line, the students were confused. They would be tempted sometimes to multiply the carry-over number. Other times they will want to add the carry-over number to the digit in the multiplier *before* multiplying. The final point we wish to emphasize is that the table should be pressed before accounting for any carry-over numbers.

The new format also highlights the two digits found in most multiplication answers. The right hand is always a column answer, the left hand a carry-over number into the next column.

Examples have been limited to three digit multipliers. However, students should find little difficulty in expanding the format into multipliers with 4 or more digits.

The first problem illustrated is 68 x 4.

HAND POSITION



Left Hand Number

Right Hand Number

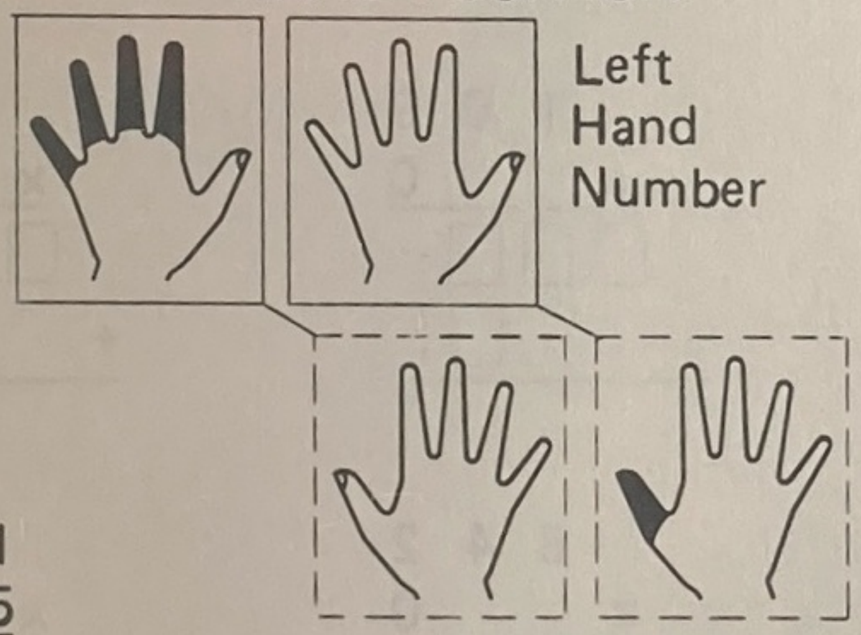
```
    6 8
  x   4
-------
  2 3
+ 4 2
-------
  2 7 2
```

The first task is to press 4 eight times. We are concentrating here on adding 4's. We therefore continue the practice, as in Stage 1, of pressing the lower multiplier repeatedly. It is most important that you press the lower number. When the students have completed that chore their fingers should read 32. They should be instructed to place the 2 from the right hand into the broken box of the units' column. The 3 on their left hand must go into the solid carry-over box in the tens' column. This instruction is easier to understand if you ask the students: "How many units on the right hand? How many tens on the left hand?"

The next task is to multiply 6 x 4. To do this you will be pressing 4 six times, but the students must know that the right hand is now counting tens and the left hand hundreds. Once they have finished pressing, ask the class: "How many tens on the right hand?" The answer, "Four" must be placed in the broken box in the tens' column. Follow this up by asking: "How many hundreds on the left hand?" The answer "Two" must be placed in the solid carry-over box in the hundreds' column. The problem is then completed by adding both rows of boxes.

The second problem, 81 x 5, is shown in order to demonstrate what to do with zeroes.

HAND POSITION




```
   8 1
x    5
 4 0
+  0 5
 4 0 5
```

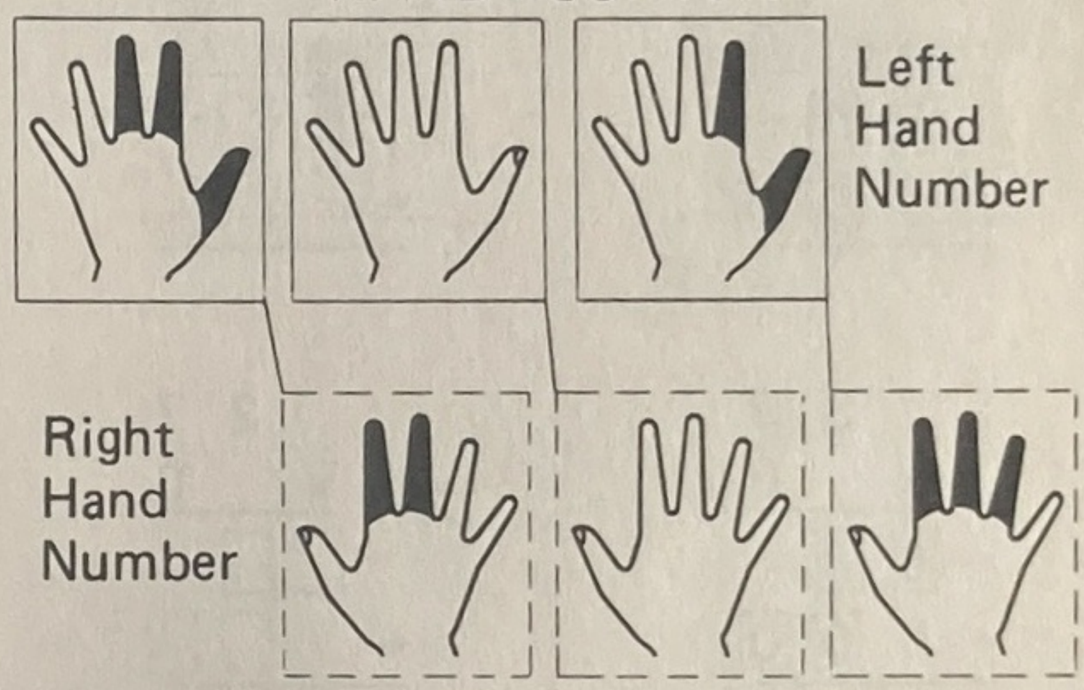
The first step in this problem is to press one 5. The number on the right hand is placed into the broken box in the units' column. The number on the left hand, (zero), *must* be placed into the solid carry-over box found in the tens' column.

The next step is to press eight 5's, again recognizing that the right hand is counting tens and the left hand is counting hundreds. When you have finished pressing, ask the class: "How many tens on the right hand?" The answer, "Zero" *must* be placed into the broken box in the tens' column. Follow this by asking: "How many hundreds on the left hand?" The answer: "Four" must be placed into the solid carry-over box in the hundreds' column. The problem is completed by adding up both rows of boxes.

The last problem to be demonstrated is 807 x 9.

```
    8 0 7
x       9
 7 0 6
+  2 0 3
 7 2 6 3
```

HAND POSITION



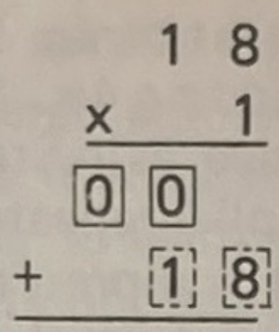

The first step is to press seven 9's. The fingers should read 63 and the 3 is placed into the broken box of the units' column and the 6 goes into the solid carry-over box of the tens' column.

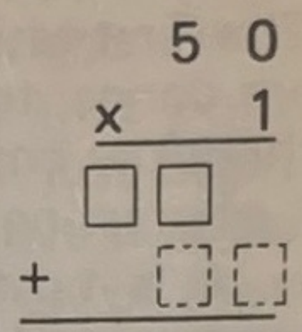
The next step would ask us to multiply 0 x 9, but this is defined as making no manipulation. The answer to this step is obviously zero. However, we have to take care to enter a zero for both hands. Therefore, a 0 is placed into the broken box of the tens' column and another zero is placed into the solid carry-over box in the hundreds' column.

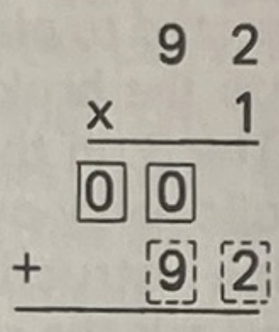
Finally, we press eight 9's recognizing that the right hand is counting hundreds and the left hand thousands. Since, at the end of pressing, you find 2 hundreds on the right hand, you place that into the broken box of the hundreds' column. The 7 on the left hand, which was counting thousands, goes into the solid carry-over box in the thousands' column. The final step is to add up both rows of boxes.

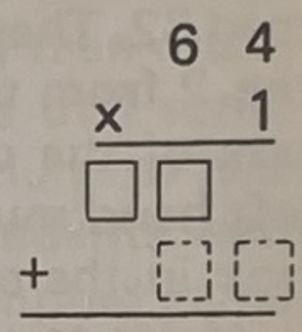
## MULTI-DIGIT MULTIPLICATION OF 0 AND 1

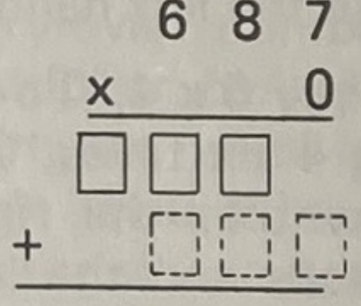
### EXERCISES

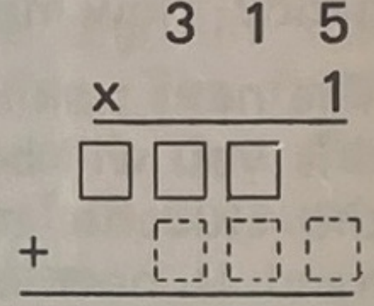


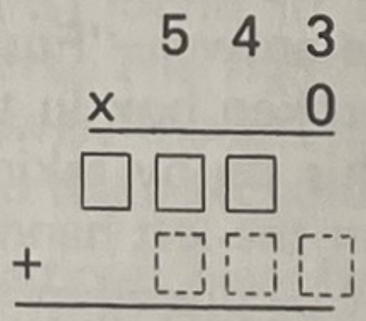


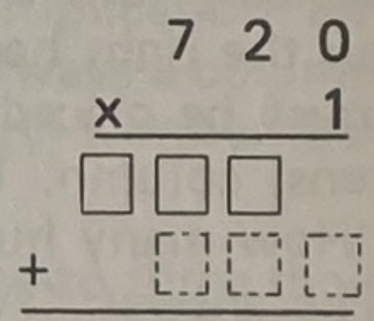


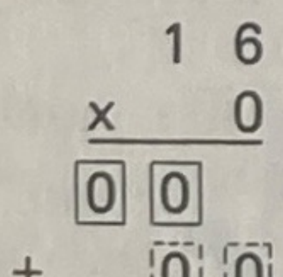


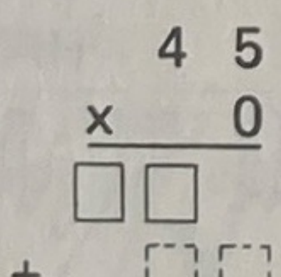


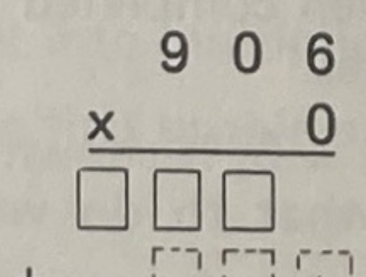


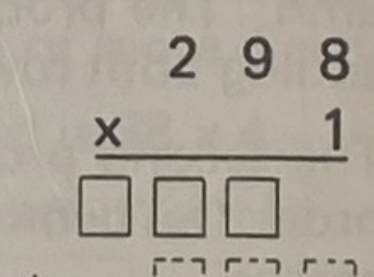


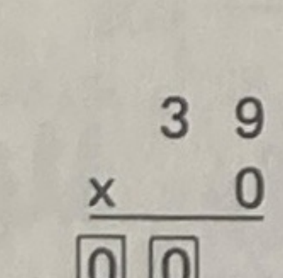


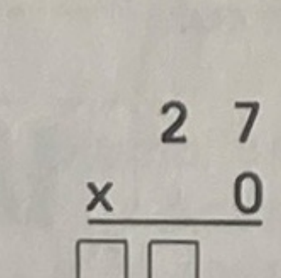


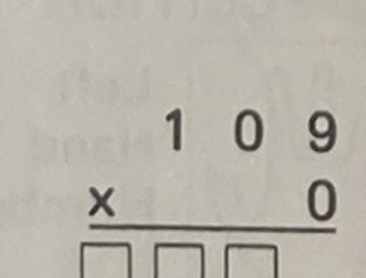


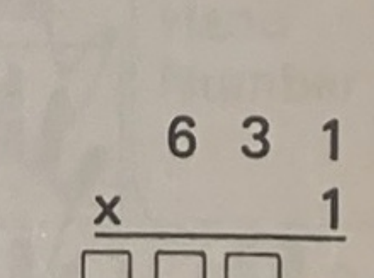


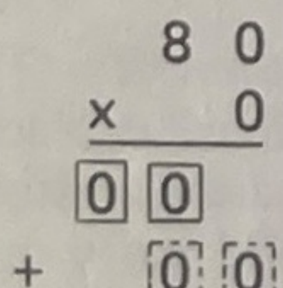


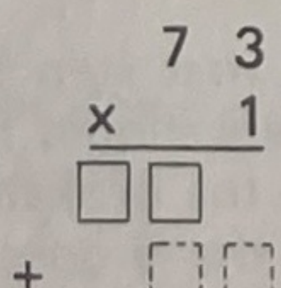


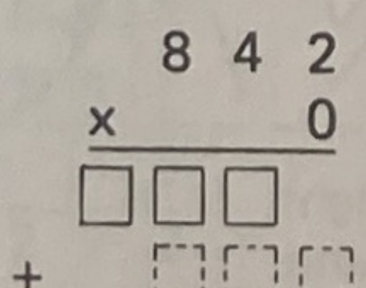


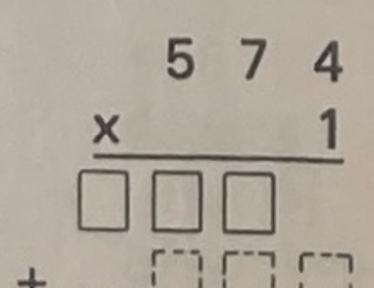










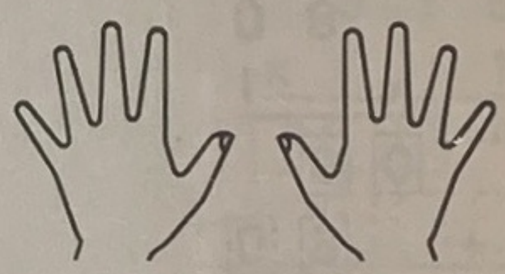
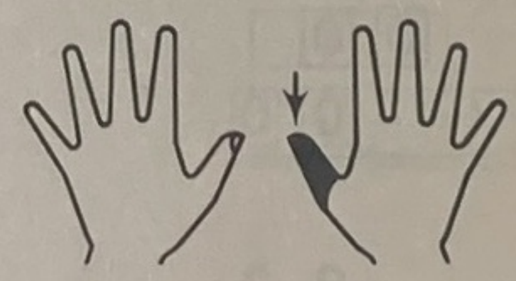
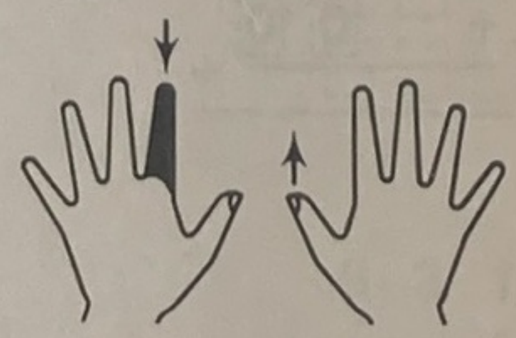
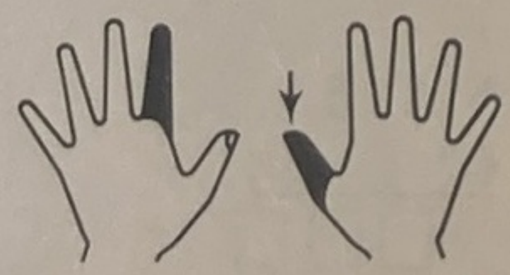


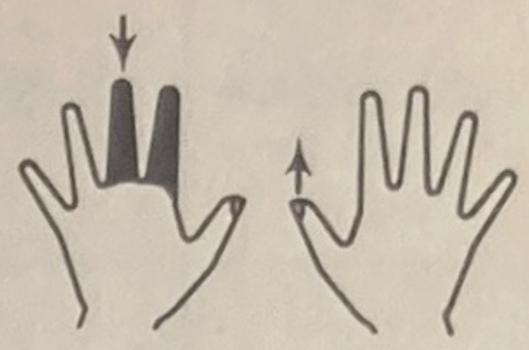
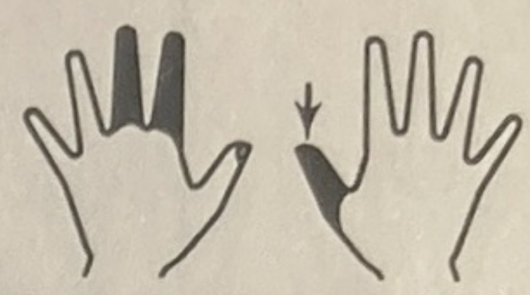
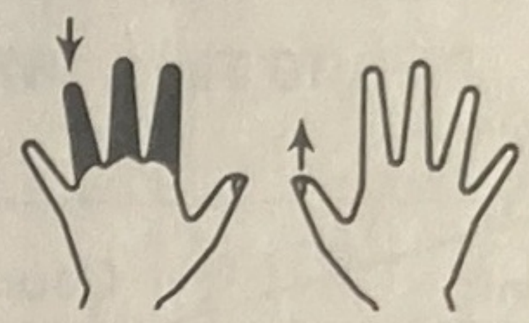
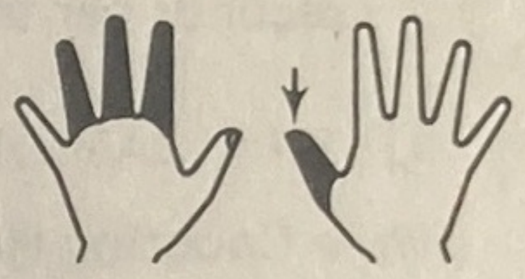
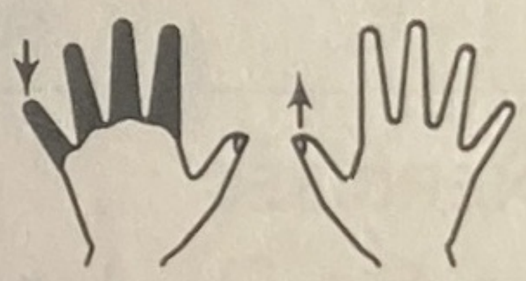
## EXERCISES

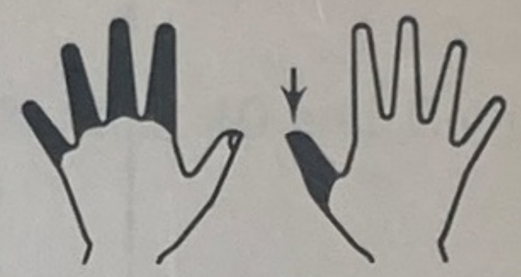
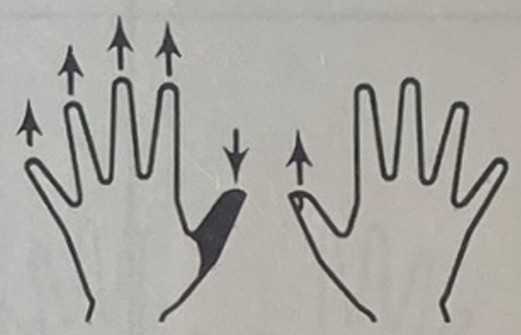
| | | | |
|---|---|---|---|
| 16<br>x 1<br>00<br>+ 16 | 45<br>x 1<br>□□<br>+ □□ | 687<br>x 1<br>□□□<br>+ □□□ | 315<br>x 0<br>□□□<br>+ □□□ |
| 39<br>x 1<br>00<br>+ 39 | 27<br>x 1<br>□□<br>+ □□ | 543<br>x 1<br>□□□<br>+ □□□ | 720<br>x 0<br>□□□<br>+ □□□ |
| 80<br>x 1<br>00<br>+ 80 | 73<br>x 0<br>□□<br>+ □□ | 906<br>x 1<br>□□□<br>+ □□□ | 298<br>x 0<br>□□□<br>+ □□□ |
| 18<br>x 0<br>00<br>+ 00 | 50<br>x 0<br>□□<br>+ □□ | 109<br>x 1<br>□□□<br>+ □□□ | 631<br>x 0<br>□□□<br>+ □□□ |
| 92<br>x 0<br>00<br>+ 00 | 64<br>x 0<br>□□<br>+ □□ | 842<br>x 1<br>□□□<br>+ □□□ | 574<br>x 0<br>□□□<br>+ □□□ |

## FIVES

All manipulations should be made simultaneously.
*Available—means fingers are available for Pressing.
**Exchange 5—means "Press 10 & Clear 5 simultaneously."
***Big Exchange 5—means "Simultaneously Left Hand—Press 50 & Clear 40 and Right Hand—Clear 5."

| Problem | Unit | Pupil Counts How Many Times | Manipulation | Fingers Read |
|---|---|---|---|---|
| 0 x 5 | – | "Zero" (Five) | No Manipulation |  0 |
| 1 x 5 | 5 | "One" (Five) | Press 5 *(Available) |  5 |
| 2 x 5 | +5 | "Two" (Fives) | **Exchange 5 (10-5) |  10 |
| 3 x 5 | +5 | "Three" (Fives) | Press 5 *(Available) |  15 |

| | | | | |
|---|---|---|---|---|
| 4 x 5 | +5 | "Four" (Fives) | **Exchange 5 (10–5) |  20 |
| 5 x 5 | +5 | "Five" (Fives) | Press 5 *(Available) |  25 |
| 6 x 5 | +5 | "Six" (Fives) | **Exchange 5 (10–5) |  30 |
| 7 x 5 | +5 | "Seven" (Fives) | Press 5 *(Available) |  35 |
| 8 x 5 | +5 | "Eight" (Fives) | **Exchange 5 (10–5) |  40 |

| | | | | |
|---|---|---|---|---|
| 9 x 5 | +5 | "Nine" (Fives) | Press 5 *(Available) |  45 |
| 10 x 5 | +5 | "Ten" (Fives) | ***Big Exchange 5 |  50 |

Total:50

(A) MULTIPLICATION STAGE 2—Time Progress Chart (A)

Press 10 Times (*Without* Counting As The Unit Is Pressed)

| Unit \ Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 50 | (time) | | | | | | | | | |

**NOTE:** You can proceed only after you have done the time progress chart (A) within 15 seconds per trial.

(B) MULTIPLICATION STAGE 2—Time Progress Chart (B)

(*While* Counting How Many Times The Unit Is Pressed)

| Unit \ Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 50 | (time) | | | | | | | | | |

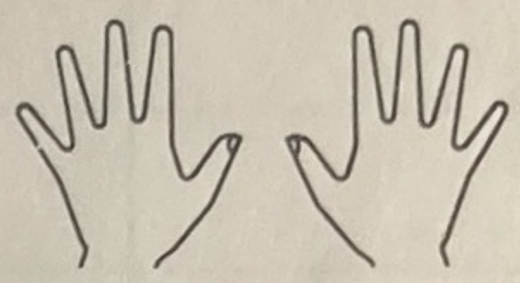
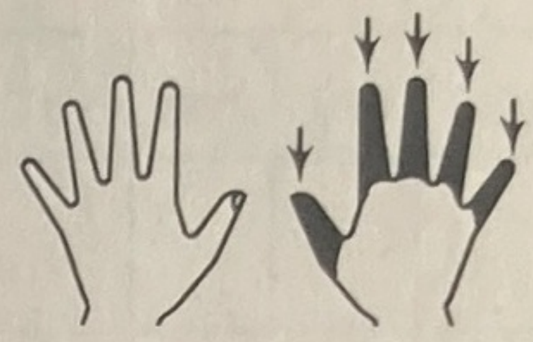
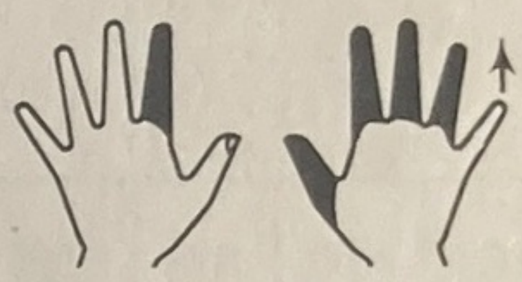
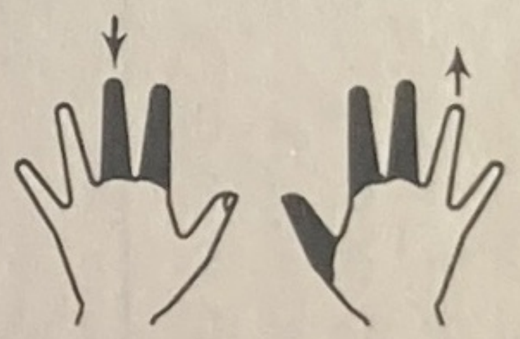
## EXERCISES

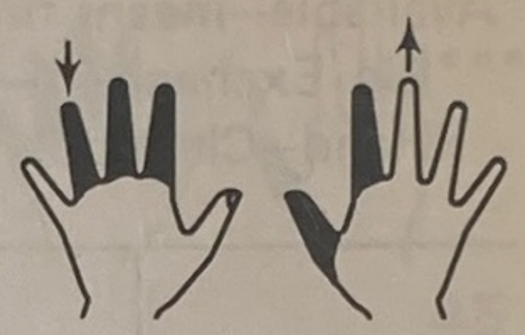
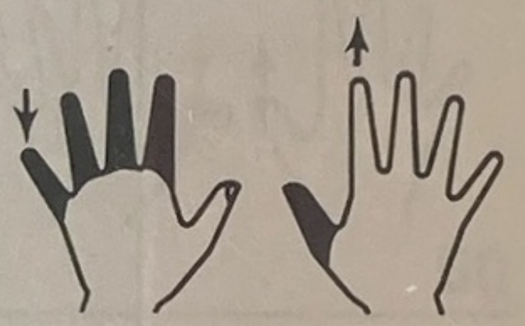
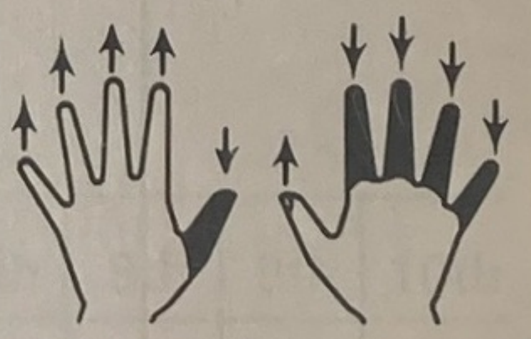
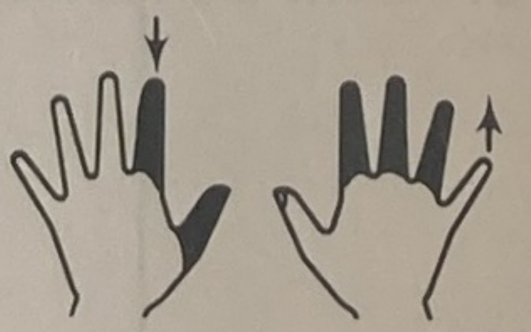
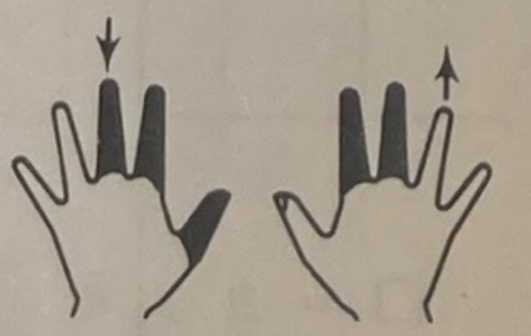
0 x 5 = ☐ 4 x 5 = ☐ 8 x 5 = ☐ 3 x 5 = ☐ 7 x 5 = ☐

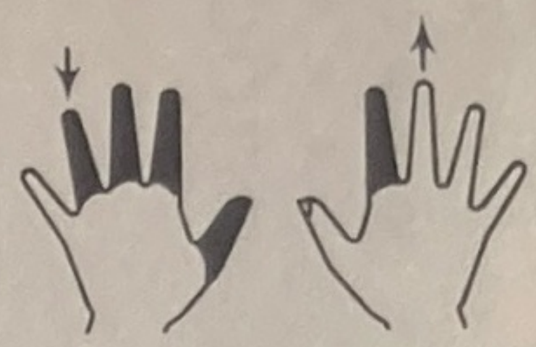
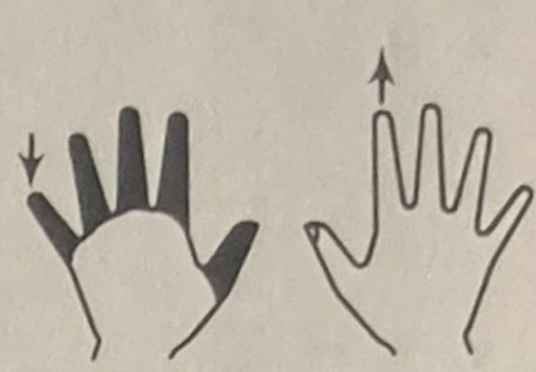
2 x 5 = ☐ 6 x 5 = ☐ 1 x 5 = ☐ 5 x 5 = ☐ 9 x 5 = ☐

## NINES

All manipulations should be made simultaneously.
*Available—means fingers are available for Pressing.
***Big Exchange 5—means "Simultaneously Left Hand—Press 50 & Clear 40 and Right Hand—Clear 5."

| Problem | Unit | Pupil Counts How Many Times | Manipulation | Fingers Read |
|---|---|---|---|---|
| 0 x 9 | — | "Zero" (Nine) | No Manipulation |  0 |
| 1 x 9 | 9 | "One" (Nine) | Press 5 & Press 4 more (5+4) *(Available) |  9 |
| 2 x 9 | +9 | "Two" (Nines) | Press 10 & Clear 1 (10–1) |  18 |
| 3 x 9 | +9 | "Three" (Nines) | Press 10 & Clear 1 (10–1) |  27 |

| | | | | |
|---|---|---|---|---|
| **4 x 9** | +9 | "Four" (Nines) | Press 10 & Clear 1 (10-1) |  **36** |
| **5 x 9** | +9 | "Five" (Nines) | Press 10 & Clear 1 (10-1) |  **45** |
| **6 x 9** | +9 | "Six" (Nines) | ***Big Exchange 5 and 4 more |  **54** |
| **7 x 9** | +9 | "Seven" (Nines) | Press 10 & Clear 1 (10-1) |  **63** |
| **8 x 9** | +9 | "Eight" (Nines) | Press 10 & Clear 1 (10-1) |  **72** |

| | | | | |
|---|---|---|---|---|
| 9 x 9 | +9 | "Nine" (Nines) | Press 10 & Clear 1 (10–1) |  81 |
| 10 x 9 | +9 | "Ten" (Nines) | Press 10 & Clear 1 (10–1) |  90 |

Total:90

(A) MULTIPLICATION STAGE 2—Time Progress Chart (A)
Press 10 Times (*Without* Counting As The Unit Is Pressed)

| Unit \ Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 90 | (time) | | | | | | | | | |

**NOTE:** You can proceed only after you have done the time progress chart (A) within 15 seconds per trial.

(B) MULTIPLICATION STAGE 2—Time Progress Chart (B)
(*While* Counting How Many Times The Unit Is Pressed)

| Unit \ Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 90 | (time) | | | | | | | | | |

## EXERCISES

| | | | |
|---|---|---|---|
| 0 x 9 = ☐ | 1 x 9 = ☐ | 6 x 9 = ☐ | 5 x 5 = ☐ |
| 2 x 9 = ☐ | 3 x 9 = ☐ | 8 x 5 = ☐ | 7 x 9 = ☐ |
| 4 x 9 = ☐ | 5 x 9 = ☐ | 3 x 9 = ☐ | 2 x 5 = ☐ |
| 6 x 9 = ☐ | 7 x 9 = ☐ | 1 x 5 = ☐ | 0 x 9 = ☐ |
| 8 x 9 = ☐ | 9 x 9 = ☐ | 4 x 9 = ☐ | 9 x 5 = ☐ |

3 x 9 = □
2 x 5 = □
7 x 9 = □
8 x 5 = □
5 x 9 = □

7 x 9 = □
3 x 5 = □
1 x 9 = □
8 x 5 = □
0 x 9 = □

4 x 9 = □
2 x 5 = □
9 x 9 = □
5 x 5 = □
3 x 9 = □

9 x 9 = □
2 x 5 = □
0 x 9 = □
1 x 5 = □
6 x 9 = □

4 x 9 = □
8 x 5 = □
1 x 9 = □
7 x 5 = □
9 x 9 = □

1 x 9 = □
6 x 5 = □
4 x 9 = □
5 x 5 = □
3 x 9 = □

6 x 9 = □
4 x 5 = □
9 x 9 = □
2 x 5 = □
0 x 9 = □

6 x 5 = □
1 x 9 = □
0 x 5 = □
9 x 9 = □
4 x 5 = □

5 x 5 = □
9 x 9 = □
2 x 5 = □
6 x 9 = □
4 x 5 = □

8 x 5 = □
0 x 9 = □
7 x 5 = □
1 x 9 = □
6 x 5 = □

2 x 9 = □
7 x 5 = □
0 x 9 = □
8 x 5 = □
3 x 5 = □

8 x 9 = □
4 x 5 = □
5 x 9 = □
7 x 5 = □
1 x 9 = □

3 x 5 = □
5 x 9 = □
7 x 5 = □
1 x 9 = □
8 x 5 = □

# MULTI-DIGIT MULTIPLICATION OF 5 AND 9 EXERCISES

```
  16
 x  5
 ---
 [0][3]
+ [5][0]
 ---
```

```
  45
 x  5
 ---
 □□
+ □□
 ---
```

```
  39
 x  5
 ---
 [1][4]
+ [5][5]
 ---
```

```
  27
 x  5
 ---
 □□
+ □□
 ---
```

```
  80
 x  5
 ---
 [4][0]
+ [0][0]
 ---
```

```
  73
 x  9
 ---
 □□
+ □□
 ---
```

```
  18
 x  9
 ---
 [0][7]
+ [9][2]
 ---
```

```
  50
 x  9
 ---
 □□
+ □□
 ---
```

```
  92
 x  9
 ---
 [8][1]
+ [1][8]
 ---
```

```
  64
 x  9
 ---
 □□
+ □□
 ---
```

```
 687
 x  5
 ---
 □□□
+ □□□
 ---
```

```
 315
 x  9
 ---
 □□□
+ □□□
 ---
```

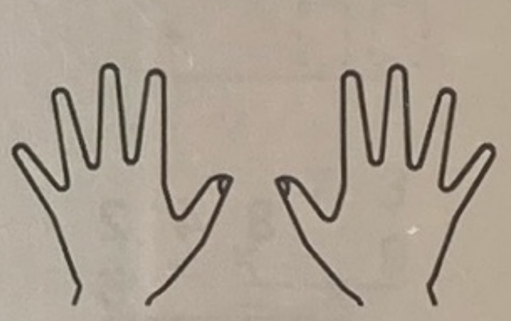
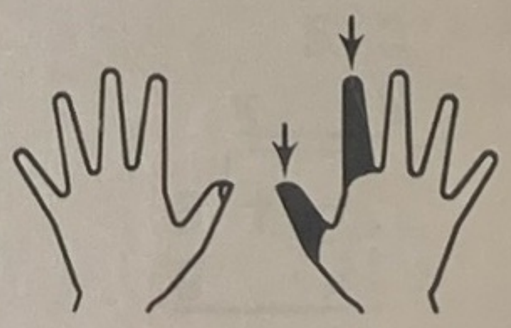
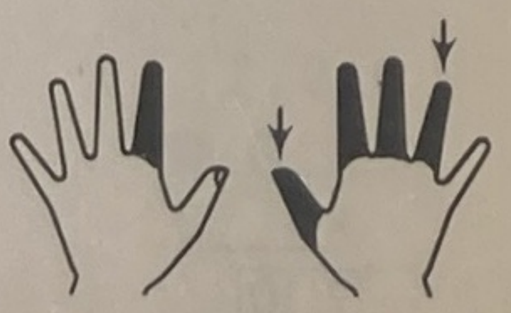
```
 543
 x  5
 ---
 □□□
+ □□□
 ---
```

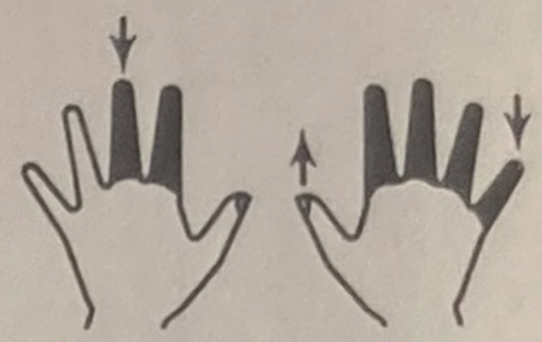
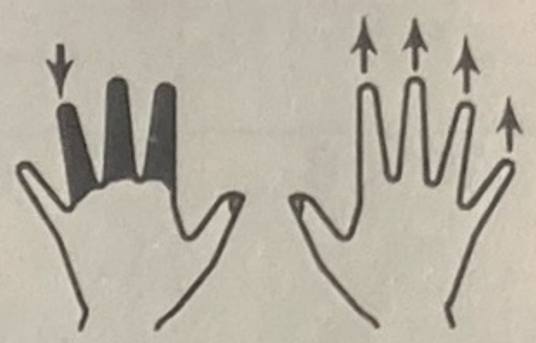
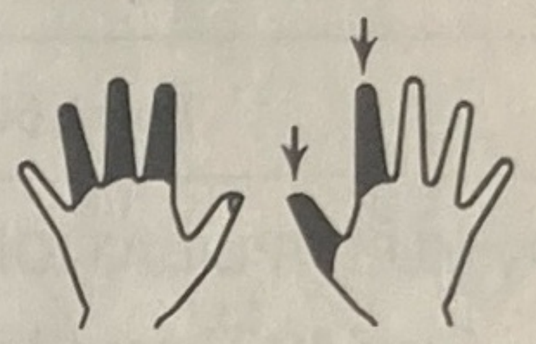
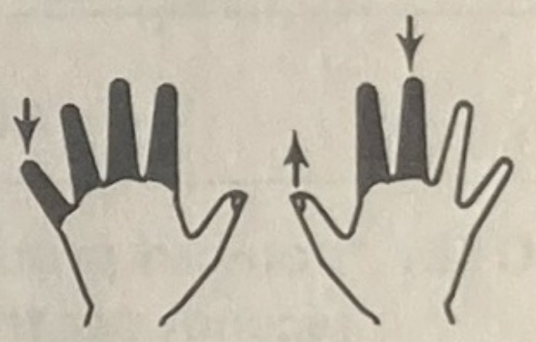
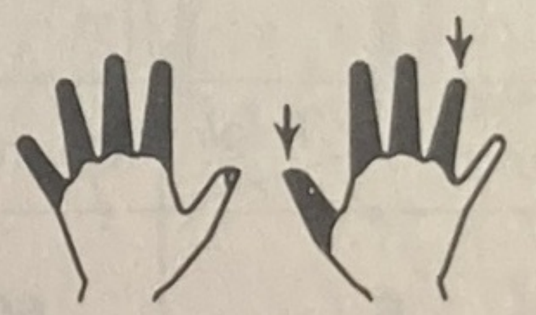
```
 720
 x  9
 ---
 □□□
+ □□□
 ---
```

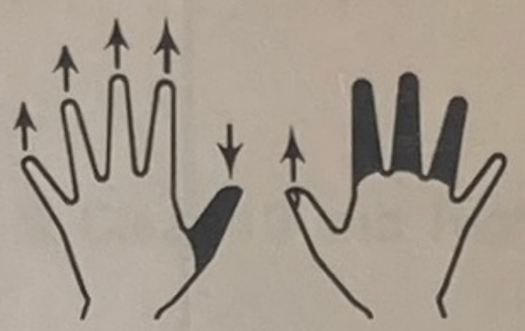
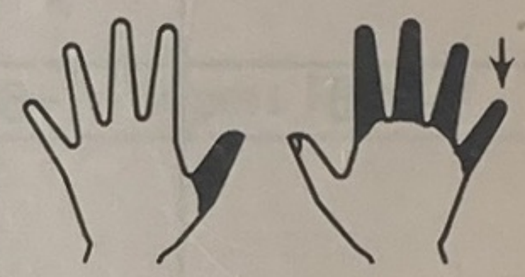
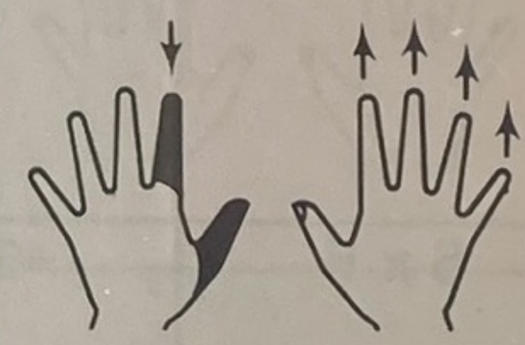
| | | | |
|---|---|---|---|
| 906<br>x 5<br>□□□<br>+ □□□ | 298<br>x 9<br>□□□<br>+ □□□ | 92<br>x 5<br>41<br>+ 50 | 64<br>x 5<br>□□<br>+ □□ |
| 109<br>x 5<br>□□□<br>+ □□□ | 631<br>x 9<br>□□□<br>+ □□□ | 687<br>x 9<br>□□□<br>+ □□□ | 315<br>x 5<br>□□□<br>+ □□□ |
| 842<br>x 5<br>□□□<br>+ □□□ | 574<br>x 9<br>□□□<br>+ □□□ | 543<br>x 9<br>□□□<br>+ □□□ | 720<br>x 5<br>□□□<br>+ □□□ |
| 16<br>x 9<br>05<br>+ 94 | 45<br>x 9<br>□□<br>+ □□ | 906<br>x 9<br>□□□<br>+ □□□ | 298<br>x 5<br>□□□<br>+ □□□ |
| 39<br>x 9<br>28<br>+ 71 | 27<br>x 9<br>□□<br>+ □□ | 109<br>x 9<br>□□□<br>+ □□□ | 631<br>x 5<br>□□□<br>+ □□□ |
| 80<br>x 9<br>70<br>+ 20 | 73<br>x 5<br>□□<br>+ □□ | 842<br>x 9<br>□□□<br>+ □□□ | 574<br>x 5<br>□□□<br>+ □□□ |

## SIXES

All manipulations are to be made simultaneously.
*Available—means "fingers are available for Pressing."
**Exchange 5—means "Press 10 & Clear 5 simultaneously."
***Big Exchange 5—means "Simultaneously Left Hand—Press 50 & Clear 40 and Right Hand—Clear 5."

| Problem | Unit | Pupil Counts How Many Times | Manipulation | Fingers Read |
|---|---|---|---|---|
| 0 x 6 | – | "Zero" (Six) | No Manipulation |  0 |
| 1 x 6 | 6 | "One" (Six) | Press 5 & Press 1 more (5+1) *(Available) |  6 |
| 2 x 6 | +6 | "Two" (Sixes) | **Exchange 5 & Press 1 more (10-5)+1 |  12 |
| 3 x 6 | +6 | "Three" (Sixes) | Press 5 & Press 1 more (5+1) *(Available) |  18 |

| | | | | |
|---|---|---|---|---|
| 4 x 6 | +6 | "Four" (Sixes) | Exchange 5 & Press 1 more (10-5)+1 |  24 |
| 5 x 6 | +6 | "Five" (Sixes) | Press 10 & Clear 4 (10-4) |  30 |
| 6 x 6 | +6 | "Six" (Sixes) | Press 5 & Press 1 more (5+1) (Available) |  36 |
| 7 x 6 | +6 | "Seven" (Sixes) | Exchange 5 & Press 1 more (10-5)+1 |  42 |
| 8 x 6 | +6 | "Eight" (Sixes) | Press 5 & Press 1 more (5+1) (Available) |  48 |

| | | | | |
|---|---|---|---|---|
| 9 x 6 | +6 | "Nine" (Sixes) | ***Big Exchange 5 and 1 more | (A)  (B)  54 |
| 10 x 6 | +6 | "Ten" (Sixes) | Press 10 & Clear 4 (10-4) |  60 |

Total:60

(A) MULTIPLICATION STAGE 2–Time Progress Chart (A)

Press 10 Times (*Without* Counting As The Unit Is Pressed)

| Unit / Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 60 | (time) | | | | | | | | | |

**NOTE:** You can proceed only after you have done the time progress chart (A) within 15 seconds per trial.

(B) MULTIPLICATION STAGE 2–Time Progress Chart (B)

(*While* Counting How Many Times The Unit Is Pressed)

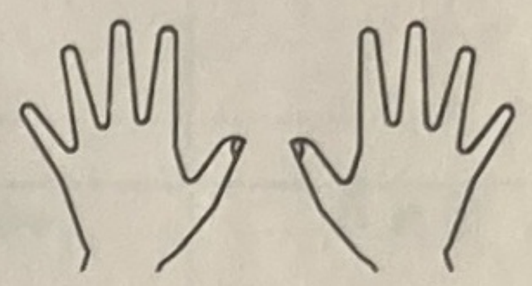
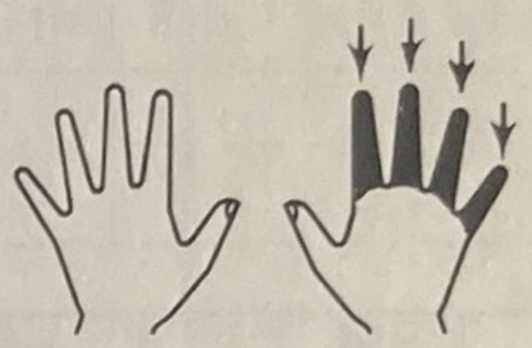
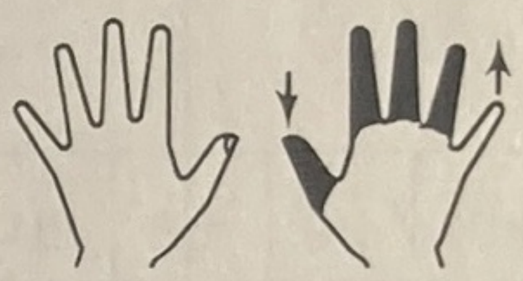
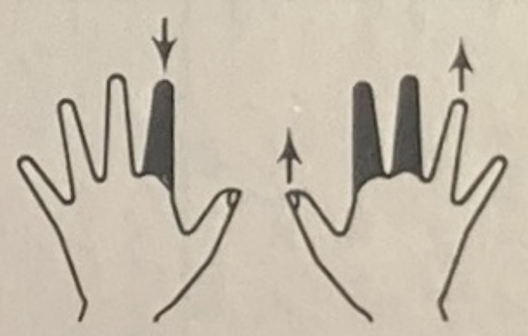
| Unit / Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 60 | (time) | | | | | | | | | |

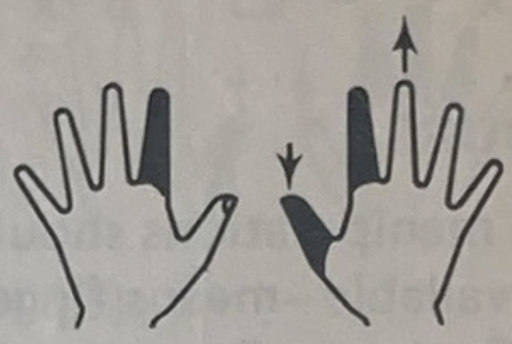
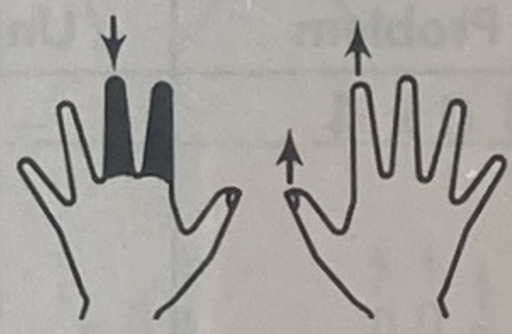
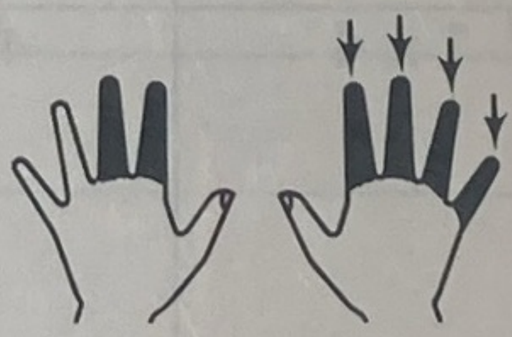
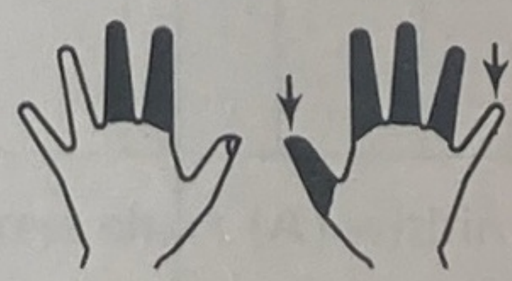
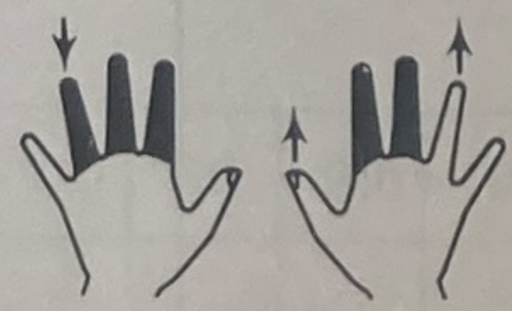
## EXERCISES

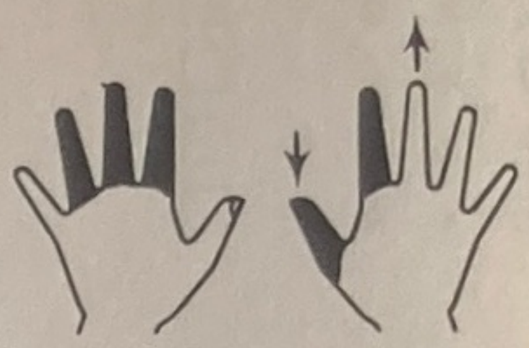
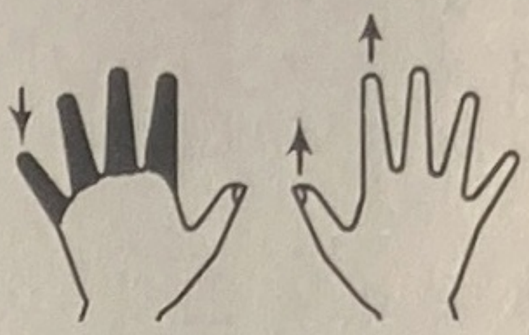
| | | | | |
|---|---|---|---|---|
| 0 x 6 = ☐ | 4 x 6 = ☐ | 8 x 6 = ☐ | 3 x 6 = ☐ | 7 x 6 = ☐ |
| 2 x 6 = ☐ | 6 x 6 = ☐ | 1 x 6 = ☐ | 5 x 6 = ☐ | 9 x 6 = ☐ |

## FOURS

All manipulations should be made simultaneously.
*Available—means fingers are available for Pressing.
**Exchange 5—means "Press 10 & Clear 5 simultaneously."

| Problem | Unit | Pupil Counts How Many Times | Manipulation | Fingers Read |
|---|---|---|---|---|
| 0 x 4 | – | "Zero" (Four) | No Manipulation |  0 |
| 1 x 4 | 4 | "One" (Four) | Press 4 *(Available) |  4 |
| 2 x 4 | +4 | "Two" (Fours) | Press 5 & Clear 1 (5–1) |  8 |
| 3 x 4 | +4 | "Three" (Fours) | **Exchange 5 & Clear 1 (10–5)–1 |  12 |

| | | | | |
|---|---|---|---|---|
| **4 x 4** | +4 | "Four" (Fours) | Press 5 & Clear 1 (5–1) |  16 |
| **5 x 4** | +4 | "Five" (Fours) | **Exchange 5 & Clear 1 (10–5)–1 |  20 |
| **6 x 4** | +4 | "Six" (Fours) | Press 4 *(Available) |  24 |
| **7 x 4** | +4 | "Seven" (Fours) | Press 5 & Clear 1 (5–1) |  28 |
| **8 x 4** | +4 | "Eight" (Fours) | **Exchange 5 & Clear 1 (10–5)–1 |  32 |

| | | | | |
|---|---|---|---|---|
| **9 x 4** | +4 | "Nine" (Fours) | Press 5 & Clear 1 (5–1) |  **36** |
| **10 x 4** | +4 | "Ten" (Fours) | **Exchange 5 & Clear 1 (10–5)–1 |  **40** |
| | Total:40 | | | |

(A) MULTIPLICATION STAGE 2—Time Progress Chart (A)

Press 10 Times (*Without* Counting As The Unit Is Pressed)

| Unit \ Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **4** | **40** | (time) | | | | | | | | | |

**NOTE:** You can proceed only after you have done the time progress chart (A) within 15 seconds per trial.

(B) MULTIPLICATION STAGE 2—Time Progress Chart (B)

(*While* Counting How Many Times The Unit Is Pressed)

| Unit \ Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **4** | **40** | (time) | | | | | | | | | |

## EXERCISES

| | | | |
|---|---|---|---|
| 0 x 4 = ☐ | 1 x 4 = ☐ | 6 x 4 = ☐ | 5 x 6 = ☐ |
| 2 x 4 = ☐ | 3 x 4 = ☐ | 8 x 6 = ☐ | 7 x 4 = ☐ |
| 4 x 4 = ☐ | 5 x 4 = ☐ | 3 x 4 = ☐ | 2 x 6 = ☐ |
| 6 x 4 = ☐ | 7 x 4 = ☐ | 1 x 6 = ☐ | 0 x 4 = ☐ |
| 8 x 4 = ☐ | 9 x 4 = ☐ | 4 x 4 = ☐ | 9 x 6 = ☐ |

3 x 4 = □
2 x 6 = □
7 x 4 = □
8 x 6 = □
5 x 4 = □

7 x 4 = □
3 x 6 = □
1 x 4 = □
8 x 6 = □
0 x 4 = □

4 x 4 = □
2 x 6 = □
9 x 4 = □
5 x 6 = □
3 x 4 = □

9 x 4 = □
2 x 6 = □
0 x 4 = □
1 x 6 = □
6 x 4 = □

4 x 4 = □
8 x 6 = □
1 x 4 = □
7 x 6 = □
9 x 4 = □

1 x 4 = □
6 x 6 = □
4 x 4 = □
5 x 6 = □
3 x 4 = □

6 x 4 = □
4 x 6 = □
9 x 4 = □
2 x 6 = □
0 x 4 = □

6 x 6 = □
1 x 4 = □
0 x 6 = □
9 x 4 = □
4 x 6 = □

5 x 6 = □
9 x 4 = □
2 x 6 = □
6 x 4 = □
4 x 6 = □

8 x 6 = □
0 x 4 = □
7 x 6 = □
1 x 4 = □
6 x 6 = □

3 x 6 = □
8 x 4 = □
4 x 6 = □
5 x 4 = □
7 x 6 = □

0 x 6 = □
3 x 4 = □
5 x 6 = □
6 x 4 = □
2 x 6 = □

7 x 6 = □
0 x 4 = □
5 x 6 = □

# MULTI-DIGIT MULTIPLICATION OF 6 AND 4 EXERCISES

$$\begin{array}{r} 16 \\ \times\ 6 \\ \hline \boxed{0}\boxed{3} \\ +\ \boxed{6}\boxed{6} \\ \hline \end{array}$$

$$\begin{array}{r} 45 \\ \times\ 6 \\ \hline \square\square \\ +\ \square\square \\ \hline \end{array}$$

$$\begin{array}{r} 39 \\ \times\ 6 \\ \hline \boxed{1}\boxed{5} \\ +\ \boxed{8}\boxed{4} \\ \hline \end{array}$$

$$\begin{array}{r} 27 \\ \times\ 6 \\ \hline \square\square \\ +\ \square\square \\ \hline \end{array}$$

$$\begin{array}{r} 80 \\ \times\ 6 \\ \hline \boxed{4}\boxed{0} \\ +\ \boxed{8}\boxed{0} \\ \hline \end{array}$$

$$\begin{array}{r} 73 \\ \times\ 4 \\ \hline \square\square \\ +\ \square\square \\ \hline \end{array}$$

$$\begin{array}{r} 18 \\ \times\ 4 \\ \hline \boxed{0}\boxed{3} \\ +\ \boxed{4}\boxed{2} \\ \hline \end{array}$$

$$\begin{array}{r} 50 \\ \times\ 4 \\ \hline \square\square \\ +\ \square\square \\ \hline \end{array}$$

$$\begin{array}{r} 92 \\ \times\ 4 \\ \hline \boxed{3}\boxed{0} \\ +\ \boxed{6}\boxed{8} \\ \hline \end{array}$$

$$\begin{array}{r} 64 \\ \times\ 4 \\ \hline \square\square \\ +\ \square\square \\ \hline \end{array}$$

$$\begin{array}{r} 687 \\ \times\ 6 \\ \hline \square\square\square \\ +\ \square\square\square \\ \hline \end{array}$$

$$\begin{array}{r} 315 \\ \times\ 4 \\ \hline \square\square\square \\ +\ \square\square\square \\ \hline \end{array}$$

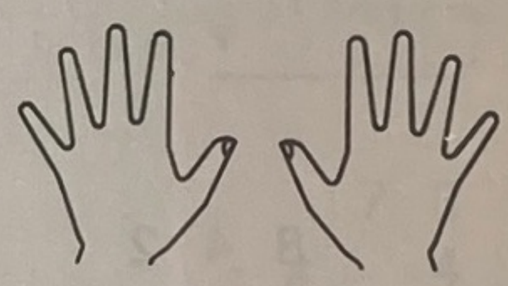
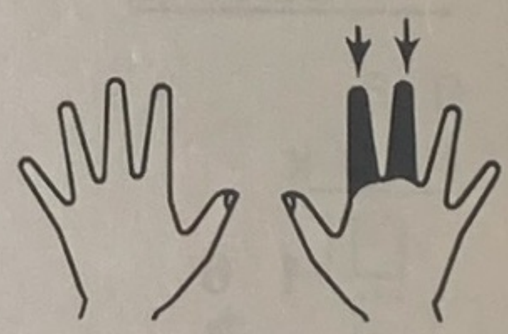
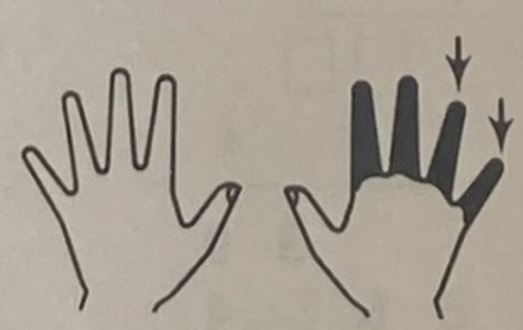
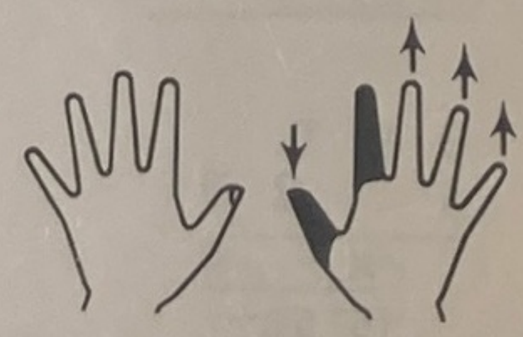
$$\begin{array}{r} 543 \\ \times\ 6 \\ \hline \square\square\square \\ +\ \square\square\square \\ \hline \end{array}$$

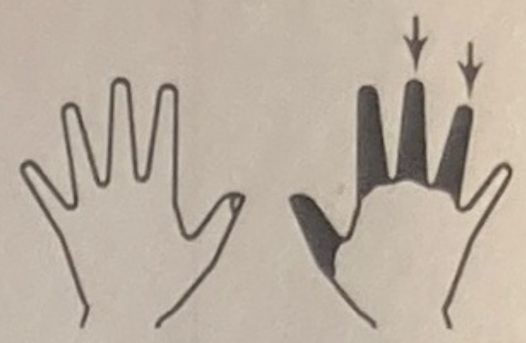
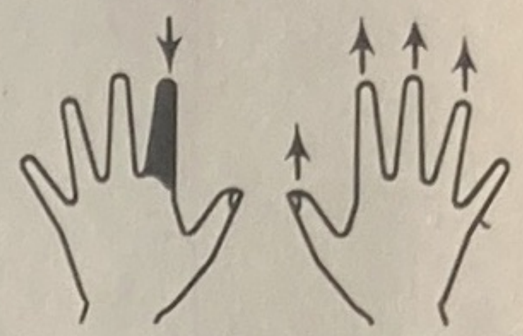
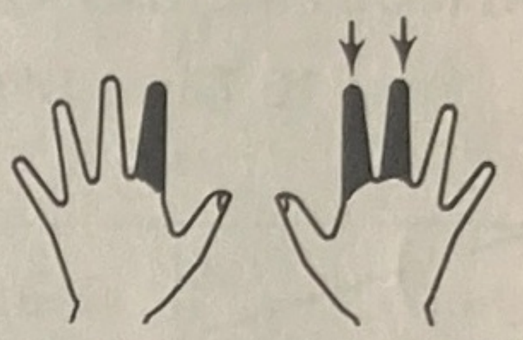
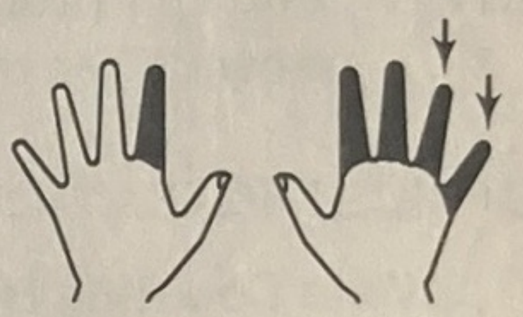
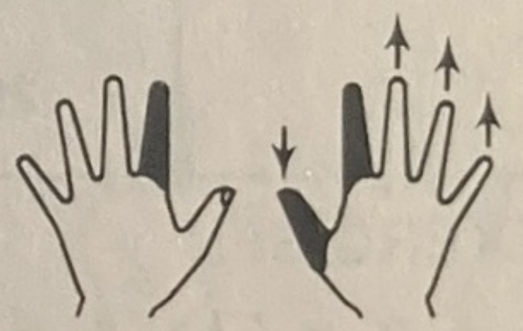
$$\begin{array}{r} 720 \\ \times\ 4 \\ \hline \square\square\square \\ +\ \square\square\square \\ \hline \end{array}$$

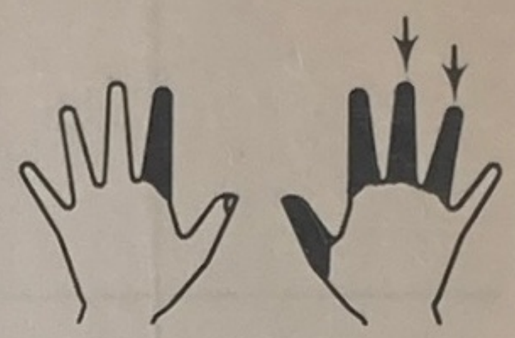
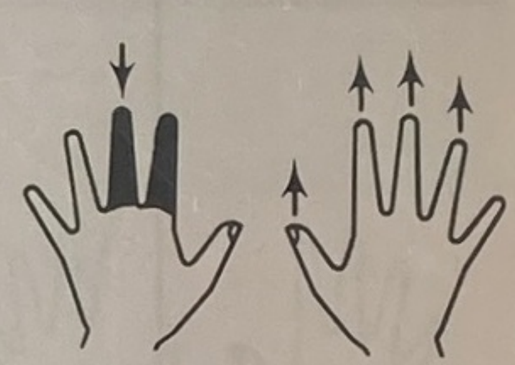
| | | | |
|---|---|---|---|
| 906<br>x 6<br>□□□<br>+ □□□ | 298<br>x 4<br>□□□<br>+ □□□ | 92<br>x 6<br>51<br>+ 42 | 64<br>x 6<br>□□<br>+ □□ |
| 109<br>x 6<br>□□□<br>+ □□□ | 631<br>x 4<br>□□□<br>+ □□□ | 687<br>x 4<br>□□□<br>+ □□□ | 315<br>x 6<br>□□□<br>+ □□□ |
| 842<br>x 6<br>□□□<br>+ □□□ | 574<br>x 4<br>□□□<br>+ □□□ | 543<br>x 4<br>□□□<br>+ □□□ | 720<br>x 6<br>□□□<br>+ □□□ |
| 16<br>x 4<br>02<br>+ 44 | 45<br>x 4<br>□□<br>+ □□ | 906<br>x 4<br>□□□<br>+ □□□ | 298<br>x 6<br>□□□<br>+ □□□ |
| 39<br>x 4<br>13<br>+ 26 | 27<br>x 4<br>□□<br>+ □□ | 109<br>x 4<br>□□□<br>+ □□□ | 631<br>x 6<br>□□□<br>+ □□□ |
| 80<br>x 4<br>30<br>+ 20 | 73<br>x 6<br>□□<br>+ □□ | 842<br>x 4<br>□□□<br>+ □□□ | 574<br>x 6<br>□□□<br>+ □□□ |

## TWOS

All manipulations should be made simultaneously.
*Available—means fingers are available for Pressing.
**Unit by Unit (Stage 1) Manipulation will simplify this transition.
***Exchange 5—means "Press 10 & Clear 5 simultaneously."

| Problem | Unit | Pupil Counts How Many Times | Manipulation | Fingers Read |
|---|---|---|---|---|
| 0 x 2 | – | "Zero" (Two) | No Manipulation |  0 |
| 1 x 2 | 2 | "One" (Two) | Press 2 *(Available) |  2 |
| 2 x 2 | +2 | "Two" (Twos) | Press 2 *(Available) |  4 |
| 3 x 2 | +2 | "Three" (Twos) | **Press 5 & Clear 3 (5-3) |  6 |

| | | | | |
|---|---|---|---|---|
| 4 x 2 | +2 | "Four" (Twos) | Press 2 *(Available) |  8 |
| 5 x 2 | +2 | "Five" (Twos) | ***Exchange 5 & Clear 3 (A) 10-5 (B) -3 |  10 |
| 6 x 2 | +2 | "Six" (Twos) | Press 2 *(Available) |  12 |
| 7 x 2 | +2 | "Seven" (Twos) | Press 2 *(Available) |  14 |
| 8 x 2 | +2 | "Eight" (Twos) | ***Press 5 & Clear 3 (5-3) |  16 |

| | | | | |
|---|---|---|---|---|
| 9 x 2 | +2 | "Nine" (Twos) | Press 2 *(Available) |  18 |
| 10 x 2 | +2 | "Ten" (Twos) | ***Exchange 5 & Clear 3 (A) 10–5) (B) –3 |  20 |
| | Total:20 | | | |

(A) MULTIPLICATION STAGE 2—Time Progress Chart (A)

Press 10 Times (*Without* Counting As The Unit Is Pressed)

| Unit \ Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 20 | (time) | | | | | | | | | |

**NOTE:** You can proceed only after you have done the time progress chart (A) within 15 seconds per trial.

(B) MULTIPLICATION STAGE 2—Time Progress Chart (B)

(*While* Counting How Many Times The Unit Is Pressed)

| Unit \ Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 20 | (time) | | | | | | | | | |

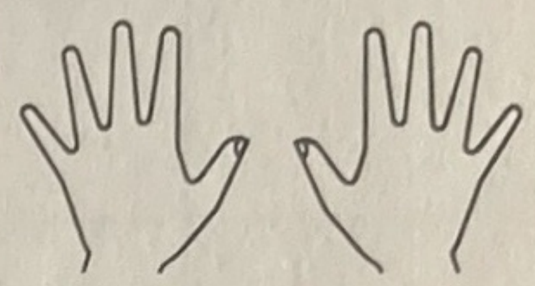
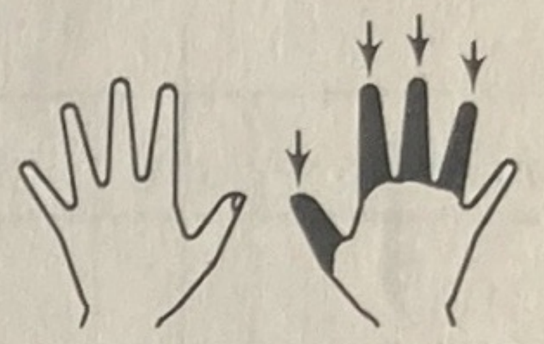
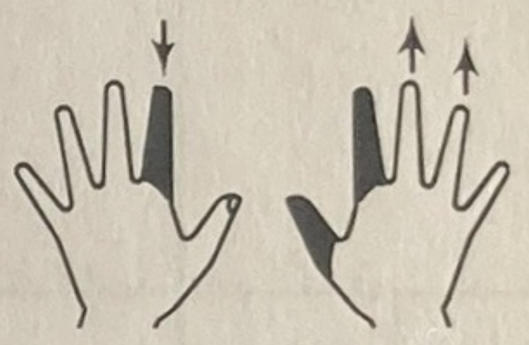
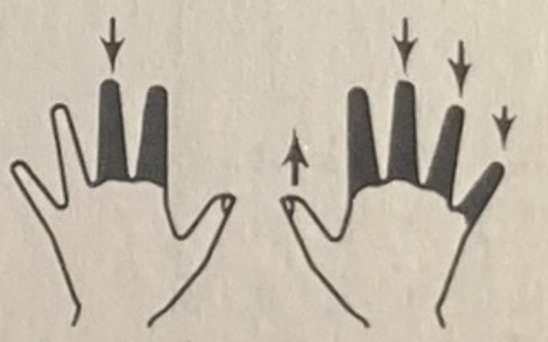
## EXERCISES

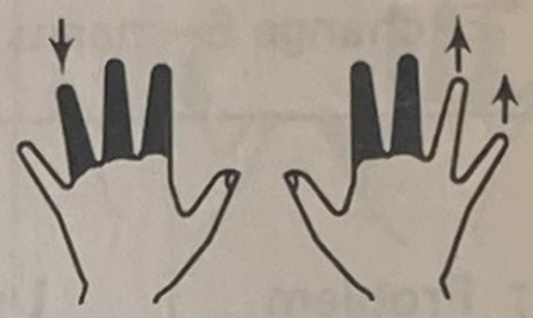
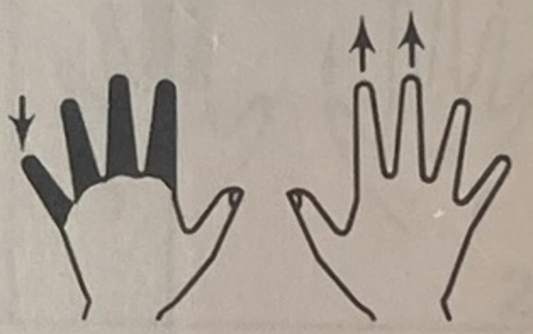
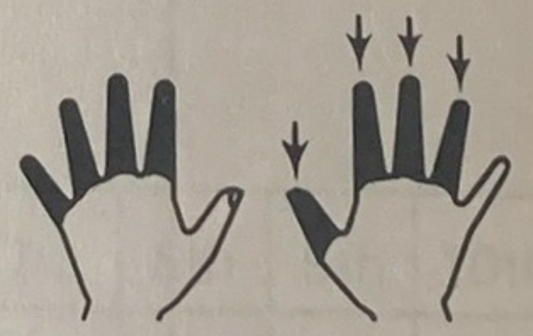
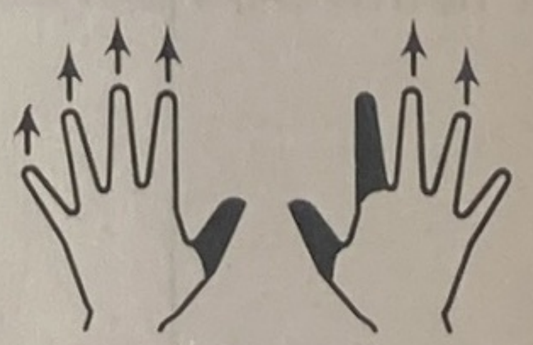
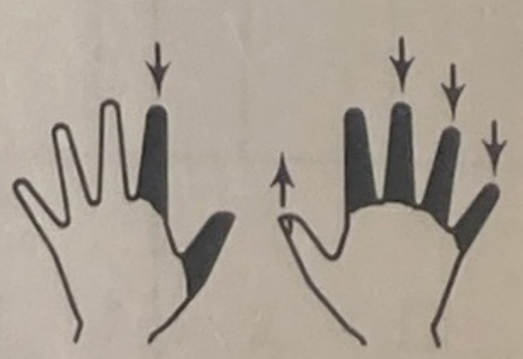
0 x 2 = ☐ 4 x 2 = ☐ 8 x 2 = ☐ 3 x 2 = ☐ 7 x 2 = ☐

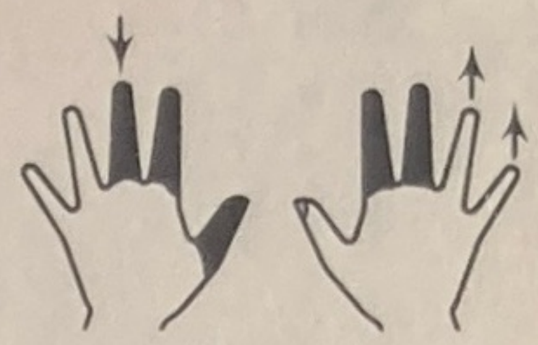
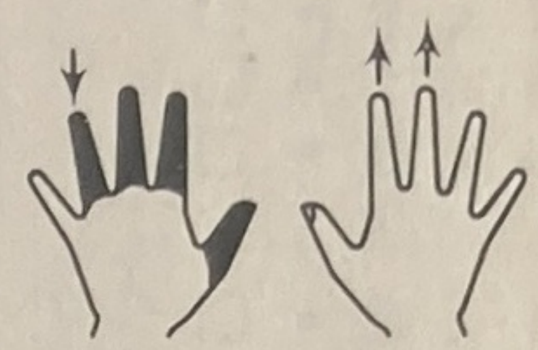
2 x 2 = ☐ 6 x 2 = ☐ 1 x 2 = ☐ 5 x 2 = ☐ 9 x 2 = ☐

## EIGHTS

All manipulations should be made simultaneously.
*Available—means fingers are available for Pressing.
**Exchange 5—means "Press 10 & Clear 5 simultaneously."

| Problem | Unit | Pupil Counts How Many Times | Manipulation | Fingers Read |
|---|---|---|---|---|
| **0 x 8** | – | "Zero" (Eight) | No Manipulation |  0 |
| **1 x 8** | +8 | "One" (Eight) | Press 5 & Press 3 more *(Available) |  8 |
| **2 x 8** | +8 | "Two" (Eights) | Press 10 & Clear 2 (10–2) |  16 |
| **3 x 8** | +8 | "Three" (Eights) | **Exchange 5 & Press 3 more (10–5)+3 |  24 |

| | | | | |
|---|---|---|---|---|
| **4 x 8** | +8 | "Four" (Eights) | Press 10 & Clear 2 (10-2) |  **32** |
| **5 x 8** | +8 | "Five" (Eights) | Press 10 & Clear 2 (10-2) |  **40** |
| **6 x 8** | +8 | "Six" (Eights) | Press 5 & Press 3 more *(Available) |  **48** |
| **7 x 8** | +8 | "Seven" (Eights) | Press 10 & Clear 2 (10-2) |  **56** |
| **8 x 8** | +8 | "Eight" (Eights) | **Exchange 5 & Press 3 more (10-5)+3 |  **64** |

| | | | | |
|---|---|---|---|---|
| **9 x 8** | +8 | "Nine" (Eights) | Press 10 & Clear 2 (10-2) |  **72** |
| **10 x 8** | +8 | "Ten" (Eights) | Press 10 & Clear 2 (10-2) |  **80** |
| | Total:80 | | | |

(A) MULTIPLICATION STAGE 2—Time Progress Chart (A)
Press 10 Times (*Without* Counting As The Unit Is Pressed)

| Unit \ Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **8** | **80** | (time) | | | | | | | | | |

**NOTE:** You can proceed only after you have done the time progress chart (A) within 15 seconds per trial.

(B) MULTIPLICATION STAGE 2—Time Progress Chart (B)
(*While* Counting How Many Times The Unit Is Pressed)

| Unit \ Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **8** | **80** | (time) | | | | | | | | | |

## EXERCISES

| | | | |
|---|---|---|---|
| 0 x 8 = ☐ | 1 x 8 = ☐ | 6 x 8 = ☐ | 5 x 2 = ☐ |
| 2 x 8 = ☐ | 3 x 8 = ☐ | 8 x 2 = ☐ | 7 x 8 = ☐ |
| 4 x 8 = ☐ | 5 x 8 = ☐ | 3 x 8 = ☐ | 2 x 2 = ☐ |
| 6 x 8 = ☐ | 7 x 8 = ☐ | 1 x 2 = ☐ | 0 x 8 = ☐ |
| 8 x 8 = ☐ | 9 x 8 = ☐ | 4 x 8 = ☐ | 9 x 2 = ☐ |

3 x 8 = □
2 x 2 = □
7 x 8 = □
8 x 2 = □
5 x 8 = □

7 x 8 = □
3 x 2 = □
1 x 8 = □
8 x 2 = □
0 x 8 = □

4 x 8 = □
2 x 2 = □
9 x 8 = □
5 x 2 = □
3 x 8 = □

9 x 8 = □
2 x 2 = □
0 x 8 = □
1 x 2 = □
6 x 8 = □

4 x 8 = □
8 x 2 = □
1 x 8 = □
7 x 2 = □
9 x 8 = □

1 x 8 = □
6 x 2 = □
4 x 8 = □
5 x 2 = □
3 x 8 = □

6 x 8 = □
4 x 2 = □
9 x 8 = □
2 x 2 = □
0 x 8 = □

6 x 2 = □
1 x 8 = □
0 x 2 = □
9 x 8 = □
4 x 2 = □

5 x 2 = □
9 x 8 = □
2 x 2 = □
6 x 8 = □
4 x 2 = □

8 x 2 = □
0 x 8 = □
7 x 2 = □
1 x 8 = □
6 x 2 = □

3 x 2 = □
8 x 8 = □
4 x 2 = □
5 x 8 = □
7 x 2 = □

0 x 2 = □
3 x 8 = □
5 x 2 = □
6 x 8 = □
2 x 2 = □

7 x 2 = □
0 x 8 = □
5 x 2 = □
3 x 2 = □
5 x 8 = □

7 x 2 = □
1 x 8 = □
8 x 2 = □

# MULTI-DIGIT MULTIPLICATION OF 2 AND 8 EXERCISES

16 x 2 = □□ + □□

45 x 2 = □□ + □□

39 x 2 = 01 + 68

27 x 2 = □□ + □□

80 x 2 = 10 + 60

73 x 8 = □□ + □□

18 x 8 = 06 + 84

50 x 8 = □□ + □□

92 x 8 = 71 + 26

64 x 8 = □□ + □□

687 x 2 = □□□ + □□□

315 x 8 = □□□ + □□□

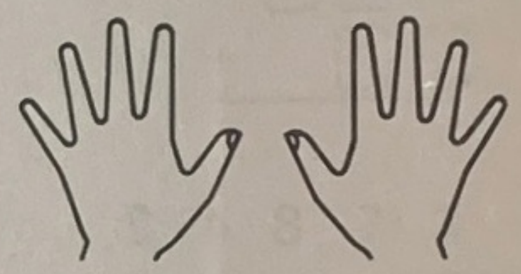
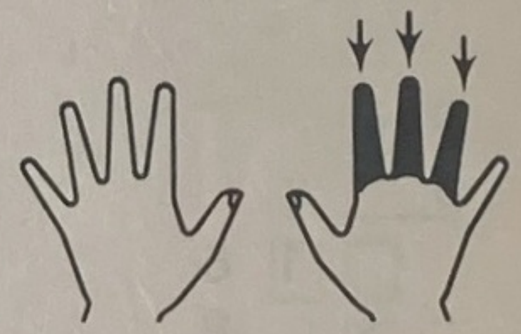
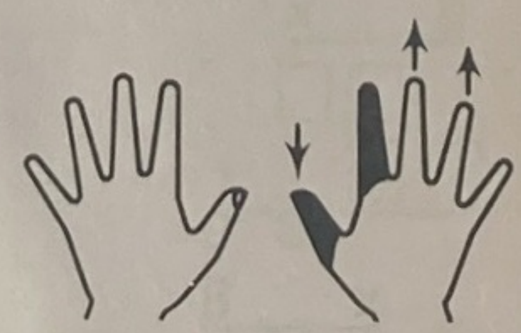
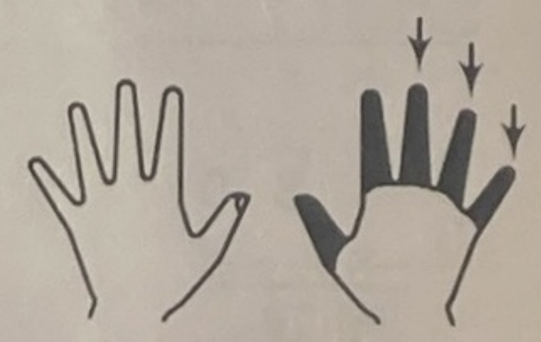
543 x 2 = □□□ + □□□

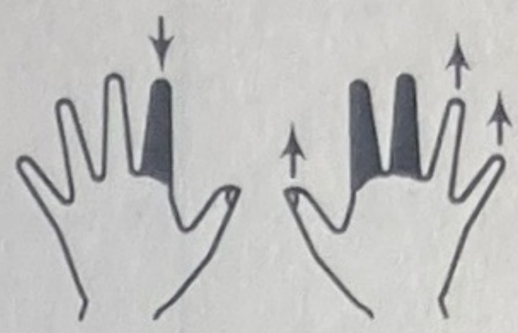
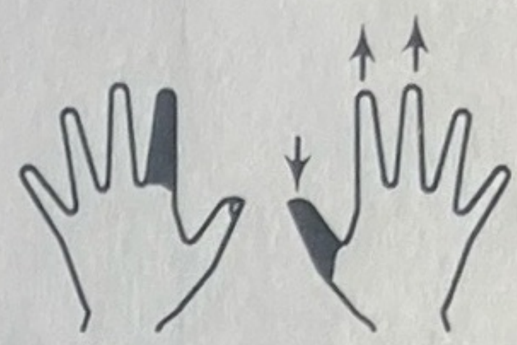
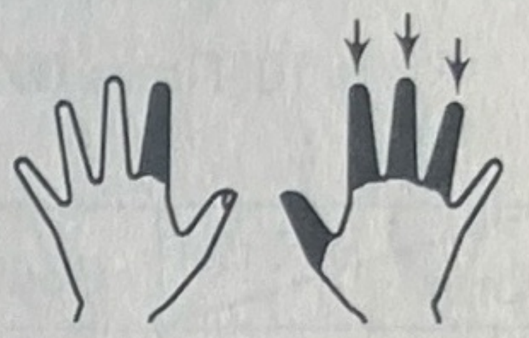
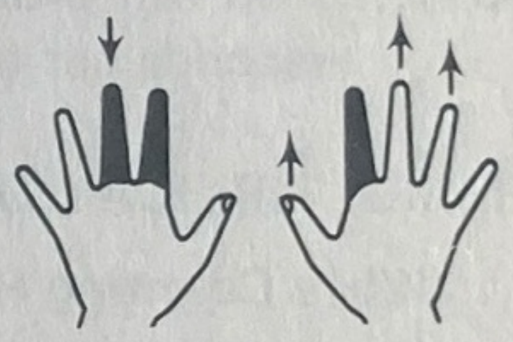
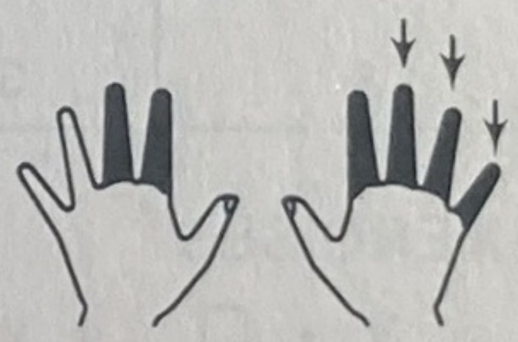
720 x 8 = □□□ + □□□

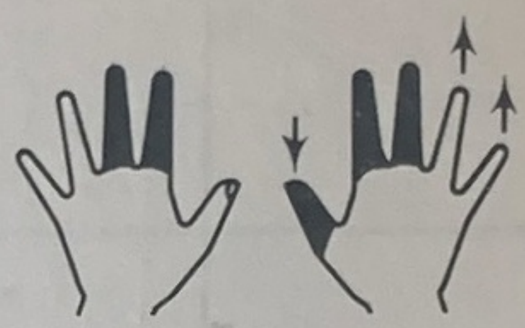
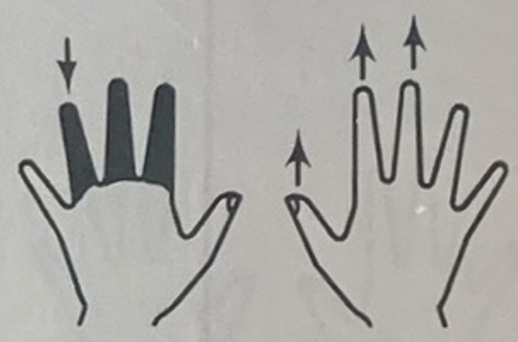
| | | | |
|---|---|---|---|
| 906<br>x 2<br>□□□<br>+ □□□ | 298<br>x 8<br>□□□<br>+ □□□ | 92<br>x 2<br>10<br>+ 84 | 64<br>x 2<br>□□<br>+ □□ |
| 109<br>x 2<br>□□□<br>+ □□□ | 631<br>x 8<br>□□□<br>+ □□□ | 687<br>x 8<br>□□□<br>+ □□□ | 315<br>x 2<br>□□□<br>+ □□□ |
| 842<br>x 2<br>□□□<br>+ □□□ | 574<br>x 8<br>□□□<br>+ □□□ | 543<br>x 8<br>□□□<br>+ □□□ | 720<br>x 2<br>□□□<br>+ □□□ |
| 16<br>x 8<br>04<br>+ 88 | 45<br>x 8<br>□□<br>+ □□ | 906<br>x 8<br>□□□<br>+ □□□ | 298<br>x 2<br>□□□<br>+ □□□ |
| 39<br>x 8<br>27<br>+ 42 | 27<br>x 8<br>□□<br>+ □□ | 109<br>x 8<br>□□□<br>+ □□□ | 631<br>x 2<br>□□□<br>+ □□□ |
| 80<br>x 8<br>60<br>+ 40 | 73<br>x 2<br>□□<br>+ □□ | 842<br>x 8<br>□□□<br>+ □□□ | 574<br>x 2<br>□□□<br>+ □□□ |

## THREES

All manipulations should be made simultaneously.
*Available—means fingers are available for Pressing.
**Exchange 5—means "Press 10 & Clear 5 simultaneously."

| Problem | Unit | Pupil Counts How Many Times | Manipulation | Fingers Read |
|---|---|---|---|---|
| 0 x 3 | – | "Zero" (Three) | No Manipulation |  0 |
| 1 x 3 | 3 | "One" (Three) | Press 3 *(Available) |  3 |
| 2 x 3 | +3 | "Two" (Threes) | Press 5 & Clear 2 (5–2) |  6 |
| 3 x 3 | +3 | "Three" (Threes) | Press 3 *(Available) |  9 |

| | | | | |
|---|---|---|---|---|
| **4 x 3** | +3 | "Four" (Threes) | **Exchange 5 & Clear 2 (10–5)–2 |  12 |
| **5 x 3** | +3 | "Five" (Threes) | Press 5 & Clear 2 (5–2) |  15 |
| **6 x 3** | +3 | "Six" (Threes) | *(Available) |  18 |
| **7 x 3** | +3 | "Seven" (Threes) | **Exchange 5 & Clear 2 (10–5)–2 |  21 |
| **8 x 3** | +3 | "Eight" (Threes) | *(Available) |  24 |

| | | | | |
|---|---|---|---|---|
| 9 x 3 | +3 | "Nine" (Threes) | Press 5 & Clear 2 (5–2) |  27 |
| 10 x 3 | +3 | "Ten" (Threes) | **Exchange 5 & Clear 2 (10–5)–2 |  30 |
| | Total:30 | | | |

(A) MULTIPLICATION STAGE 2—Time Progress Chart (A)

Press 10 Times (*Without* Counting As The Unit Is Pressed)

| Unit \ Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 30 | (time) | | | | | | | | | |

**NOTE:** You can proceed only after you have done the time progress chart (A) within 15 seconds per trial.

(B) MULTIPLICATION STAGE 2—Time Progress Chart (B)

(*While* Counting How Many Times The Unit Is Pressed)

| Unit \ Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 30 | (time) | | | | | | | | | |

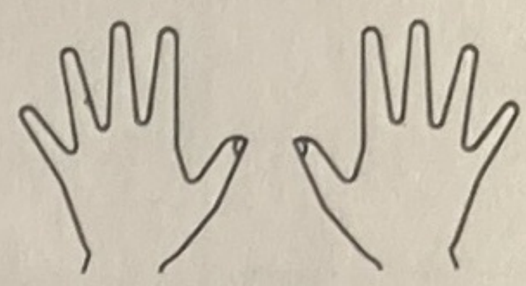
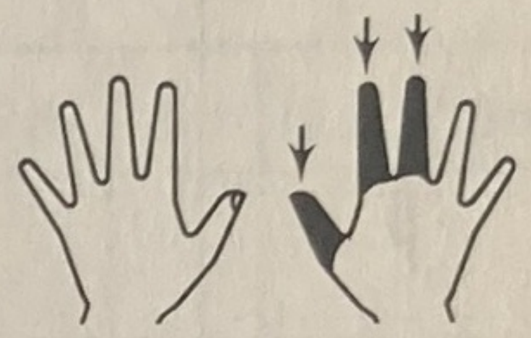
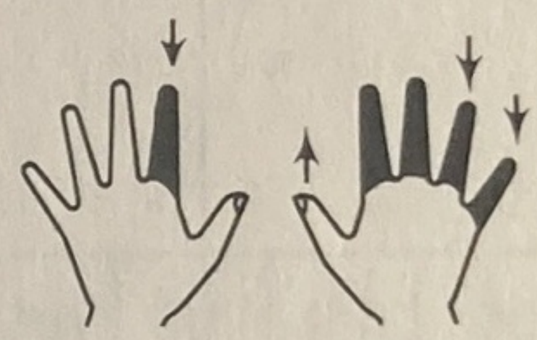
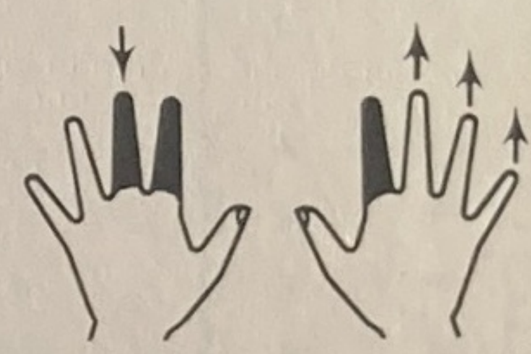
## EXERCISES

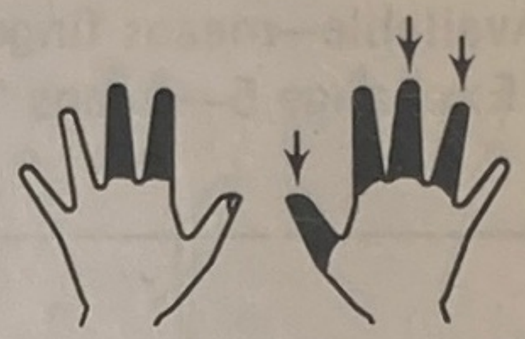
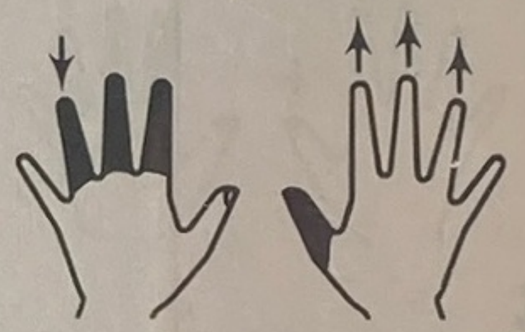
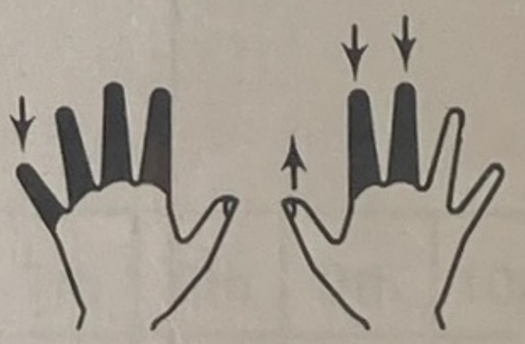
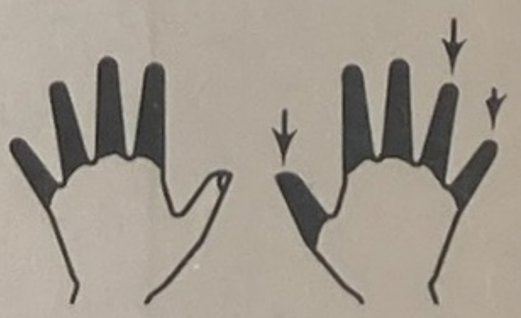
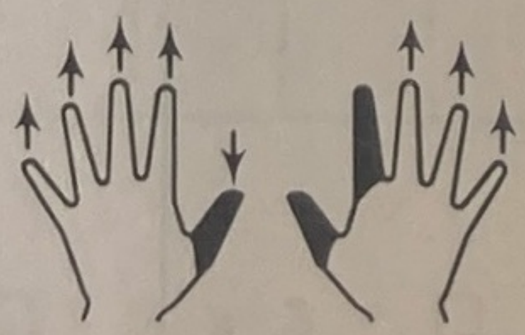
0 x 3 = ☐ 4 x 3 = ☐ 8 x 3 = ☐ 3 x 3 = ☐ 7 x 3 = ☐

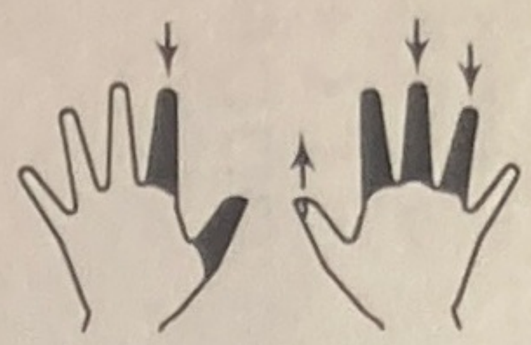
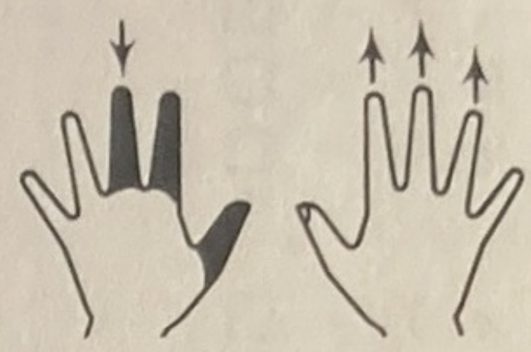
2 x 3 = ☐ 6 x 3 = ☐ 1 x 3 = ☐ 5 x 3 = ☐ 9 x 3 = ☐

## SEVENS

All manipulations should be made simultaneously.
*Available—means fingers are available for Pressing.
**Exchange 5—means "Press 10 & Clear 5 simultaneously."

| Problem | Unit | Pupil Counts How Many Times | Manipulation | Fingers Read |
|---|---|---|---|---|
| **0 x 7** | – | "Zero" (Seven) | No Manipulation |  **0** |
| **1 x 7** | 7 | "One" (Seven) | Press 5 & Press 2 more (5+2) *(Available) |  **7** |
| **2 x 7** | +7 | "Two" (Sevens) | **Exchange 5 & Press 2 more (10–5)+2 |  **14** |
| **3 x 7** | +7 | "Three" (Sevens) | Press 10 & Clear 3 (10–3) |  **21** |

| | | | | |
|---|---|---|---|---|
| **4 x 7** | +7 | "Four" (Sevens) | Press 5 & Press 2 more (5+2) *(Available) |  **28** |
| **5 x 7** | +7 | "Five" (Sevens) | Press 10 & Clear 3 (10–3) |  **35** |
| **6 x 7** | +7 | "Six" (Sevens) | **Exchange 5 & Press 2 more (10–5)+2 |  **42** |
| **7 x 7** | +7 | "Seven" (Sevens) | Press 5 & Press 2 more (5+2) *(Available) |  **49** |
| **8 x 7** | +7 | "Eight" (Sevens) | Press 10 & Clear 3 (10–3) |  **56** |

| | | | | |
|---|---|---|---|---|
| **9 x 7** | +7 | "Nine" (Sevens) | **Exchange 5 & Press 2 more (10–5)+2 |  **63** |
| **10 x 7** | +7 | "Ten" (Sevens) | Press 10 & Clear 3 (10–3) |  **70** |

Total:70

(A) MULTIPLICATION STAGE 2—Time Progress Chart (A)
Press 10 Times (*Without* Counting As The Unit Is Pressed)

| Unit \ Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **7** | **70** | (time) | | | | | | | | | |

**NOTE:** You can proceed only after you have done the time progress chart (A) within 15 seconds per trial.

(B) MULTIPLICATION STAGE 2—Time Progress Chart (B)
(*While* Counting How Many Times The Unit Is Pressed)

| Unit \ Trial | Count to | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **7** | **70** | (time) | | | | | | | | | |

## EXERCISES

| | | | |
|---|---|---|---|
| 0 x 7 = ☐ | 1 x 7 = ☐ | 6 x 7 = ☐ | 5 x 3 = ☐ |
| 2 x 7 = ☐ | 3 x 7 = ☐ | 8 x 3 = ☐ | 7 x 7 = ☐ |
| 4 x 7 = ☐ | 5 x 7 = ☐ | 3 x 7 = ☐ | 2 x 3 = ☐ |
| 6 x 7 = ☐ | 7 x 7 = ☐ | 1 x 3 = ☐ | 0 x 7 = ☐ |
| 8 x 7 = ☐ | 9 x 7 = ☐ | 4 x 7 = ☐ | 9 x 3 = ☐ |

3 x 7 = □
2 x 3 = □
7 x 7 = □
8 x 3 = □
5 x 7 = □

7 x 7 = □
3 x 3 = □
1 x 7 = □
8 x 3 = □
0 x 7 = □

4 x 7 = □
2 x 3 = □
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9 x 7 = □
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0 x 7 = □

6 x 3 = □
1 x 7 = □
0 x 3 = □
9 x 7 = □
4 x 3 = □

5 x 3 = □
9 x 7 = □
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6 x 7 = □
4 x 3 = □

8 x 3 = □
0 x 7 = □
7 x 3 = □
1 x 7 = □
6 x 3 = □

3 x 3 = □
8 x 7 = □
4 x 3 = □
5 x 7 = □
7 x 3 = □

0 x 3 = □
3 x 7 = □
5 x 3 = □
6 x 7 = □
2 x 3 = □

3 x 3 = □
5 x 7 = □
7 x 3 = □
1 x 7 = □
8 x 3 = □

# MULTI-DIGIT MULTIPLICATION OF 3 AND 7 EXERCISES

$$\begin{array}{r} 16 \\ \times\ 3 \\ \hline \boxed{0}\boxed{1} \\ +\ \boxed{3}\boxed{8} \\ \hline \end{array}$$

$$\begin{array}{r} 45 \\ \times\ 3 \\ \hline \square\square \\ +\ \square\square \\ \hline \end{array}$$

$$\begin{array}{r} 39 \\ \times\ 3 \\ \hline \boxed{0}\boxed{2} \\ +\ \boxed{9}\boxed{7} \\ \hline \end{array}$$

$$\begin{array}{r} 27 \\ \times\ 3 \\ \hline \square\square \\ +\ \square\square \\ \hline \end{array}$$

$$\begin{array}{r} 80 \\ \times\ 3 \\ \hline \boxed{2}\boxed{0} \\ +\ \boxed{4}\boxed{0} \\ \hline \end{array}$$

$$\begin{array}{r} 73 \\ \times\ 7 \\ \hline \square\square \\ +\ \square\square \\ \hline \end{array}$$

$$\begin{array}{r} 18 \\ \times\ 7 \\ \hline \boxed{0}\boxed{5} \\ +\ \boxed{7}\boxed{6} \\ \hline \end{array}$$

$$\begin{array}{r} 50 \\ \times\ 7 \\ \hline \square\square \\ +\ \square\square \\ \hline \end{array}$$

$$\begin{array}{r} 92 \\ \times\ 7 \\ \hline \boxed{6}\boxed{1} \\ +\ \boxed{3}\boxed{4} \\ \hline \end{array}$$

$$\begin{array}{r} 64 \\ \times\ 7 \\ \hline \square\square \\ +\ \square\square \\ \hline \end{array}$$

$$\begin{array}{r} 687 \\ \times\ 3 \\ \hline \square\square\square \\ +\ \square\square\square \\ \hline \end{array}$$

$$\begin{array}{r} 315 \\ \times\ 7 \\ \hline \square\square\square \\ +\ \square\square\square \\ \hline \end{array}$$

$$\begin{array}{r} 543 \\ \times\ 3 \\ \hline \square\square\square \\ +\ \square\square\square \\ \hline \end{array}$$

$$\begin{array}{r} 720 \\ \times\ 7 \\ \hline \square\square\square \\ +\ \square\square\square \\ \hline \end{array}$$

| | | | |
|---|---|---|---|
| 906<br>x 3<br>□□□<br>+ □□□ | 298<br>x 7<br>□□□<br>+ □□□ | 92<br>x 3<br>20<br>+ 76 | 64<br>x 3<br>□□<br>+ □□ |
| 109<br>x 3<br>□□□<br>+ □□□ | 631<br>x 7<br>□□□<br>+ □□□ | 687<br>x 7<br>□□□<br>+ □□□ | 315<br>x 3<br>□□□<br>+ □□□ |
| 842<br>x 3<br>□□□<br>+ □□□ | 574<br>x 7<br>□□□<br>+ □□□ | 543<br>x 7<br>□□□<br>+ □□□ | 720<br>x 3<br>□□□<br>+ □□□ |
| 16<br>x 7<br>04<br>+ 72 | 45<br>x 7<br>□□<br>+ □□ | 906<br>x 7<br>□□□<br>+ □□□ | 298<br>x 3<br>□□□<br>+ □□□ |
| 39<br>x 7<br>26<br>+ 13 | 27<br>x 7<br>□□<br>+ □□ | 109<br>x 7<br>□□□<br>+ □□□ | 631<br>x 3<br>□□□<br>+ □□□ |
| 80<br>x 7<br>50<br>+ 60 | 73<br>x 3<br>□□<br>+ □□ | 842<br>x 7<br>□□□<br>+ □□□ | 574<br>x 3<br>□□□<br>+ □□□ |

## MIXED MULTIPLIERS EXERCISES

6 x 7 = ☐
8 x 5 = ☐
3 x 0 = ☐
1 x 1 = ☐
4 x 6 = ☐

3 x 3 = ☐
2 x 2 = ☐
7 x 8 = ☐
8 x 9 = ☐
5 x 4 = ☐

7 x 3 = ☐
3 x 6 = ☐
1 x 2 = ☐
8 x 5 = ☐
0 x 8 = ☐

4 x 6 = ☐
2 x 3 = ☐
9 x 9 = ☐
5 x 0 = ☐
3 x 8 = ☐

9 x 6 = ☐
2 x 3 = ☐
0 x 1 = ☐
1 x 8 = ☐
6 x 4 = ☐

4 x 4 = ☐
8 x 3 = ☐
1 x 8 = ☐
7 x 1 = ☐
9 x 6 = ☐

1 x 8 = ☐
6 x 0 = ☐
4 x 3 = ☐
5 x 7 = ☐
3 x 4 = ☐

5 x 9 = ☐
7 x 2 = ☐
2 x 4 = ☐
0 x 8 = ☐
9 x 3 = ☐

6 x 6 = ☐
1 x 5 = ☐
0 x 1 = ☐
9 x 0 = ☐
4 x 7 = ☐

5 x 1 = ☐
9 x 9 = ☐
2 x 0 = ☐
6 x 4 = ☐
4 x 7 = ☐

8 x 2 = ☐
0 x 1 = ☐
7 x 4 = ☐
1 x 7 = ☐
6 x 5 = ☐

3 x 0 = ☐
8 x 2 = ☐
4 x 5 = ☐
5 x 9 = ☐
7 x 7 = ☐

0 x 7 = ☐
3 x 0 = ☐
5 x 5 = ☐
6 x 9 = ☐
2 x 2 = ☐

9 x 5 = ☐
2 x 1 = ☐
7 x 9 = ☐
0 x 6 = ☐
5 x 2 = ☐

6 x 3 = ☐
4 x 6 = ☐
9 x 9 = ☐
2 x 0 = ☐
0 x 7 = ☐

3 x 5 = ☐
5 x 8 = ☐
7 x 1 = ☐
1 x 4 = ☐
8 x 2 = ☐

## MULTI-DIGIT MULTIPLICATION OF MIXED NUMBERS EXERCISES

```
  16
 x  5
 ----
 □□
+ □□
 ----
```

```
  45
 x  7
 ----
 □□
+ □□
 ----
```

```
  39
 x  0
 ----
 □□
+ □□
 ----
```

```
  27
 x  3
 ----
 □□
+ □□
 ----
```

```
  80
 x  6
 ----
 □□
+ □□
 ----
```

```
  73
 x  4
 ----
 □□
+ □□
 ----
```

```
  18
 x  2
 ----
 □□
+ □□
 ----
```

```
  50
 x  8
 ----
 □□
+ □□
 ----
```

```
  92
 x  9
 ----
 □□
+ □□
 ----
```

```
  64
 x  1
 ----
 □□
+ □□
 ----
```

| | | | |
|---|---|---|---|
| 687<br>x 5<br>□□□<br>+ □□□ | 315<br>x 7<br>□□□<br>+ □□□ | 80<br>x 4<br>□□<br>+ □□ | 73<br>x 6<br>□□<br>+ □□ |
| 543<br>x 0<br>□□□<br>+ □□□ | 720<br>x 3<br>□□□<br>+ □□□ | 18<br>x 8<br>□□<br>+ □□ | 50<br>x 2<br>□□<br>+ □□ |
| 906<br>x 6<br>□□□<br>+ □□□ | 298<br>x 4<br>□□□<br>+ □□□ | 92<br>x 1<br>□□<br>+ □□ | 64<br>x 9<br>□□<br>+ □□ |
| 109<br>x 2<br>□□□<br>+ □□□ | 631<br>x 8<br>□□□<br>+ □□□ | 687<br>x 7<br>□□□<br>+ □□□ | 315<br>x 5<br>□□□<br>+ □□□ |
| 842<br>x 9<br>□□□<br>+ □□□ | 574<br>x 1<br>□□□<br>+ □□□ | 543<br>x 3<br>□□□<br>+ □□□ | 720<br>x 0<br>□□□<br>+ □□□ |
| 16<br>x 7<br>□□<br>+ □□ | 45<br>x 5<br>□□<br>+ □□ | 906<br>x 4<br>□□□<br>+ □□□ | 298<br>x 6<br>□□□<br>+ □□□ |
| 39<br>x 3<br>□□<br>+ □□ | 27<br>x 0<br>□□<br>+ □□ | 109<br>x 8<br>□□□<br>+ □□□ | 631<br>x 2<br>□□□<br>+ □□□ |

## ADDITION—STAGE 3

Addition Stage 3 presents another startling opportunity for increasing the speed of calculation. After completion of Stage 3, the calculations should be at least twice as rapid as they were at the end of Stage 2. Stage 3 also concentrates on the constant renaming of numbers thus reenforcing a mathematical concept first presented in Addition Stage 2.

The key to adding in Stage 3 is the simultaneous pressing of any single unit. How to press this number simultaneously will, of course, depend on what fingers have already been pressed on the right hand.

To show how a number should be pressed using 3rd stage manipulations, two examples are given (the units 3 & 8). These examples are followed by a pattern chart which demonstrates the patterns for all the units. Finally the chart is followed by many problems to give the reader plenty of practice in Addition Stage 3.

### Points to Keep in Mind

Remember that in the beginning of the 3rd stage addition pupils require considerable oral practice with the teacher giving oral clues before each manipulation. As the child performs each manipulation he should repeat the clues the teacher has given.

EXAMPLE A: (Adding the number 3)

There are three different ways of adding 3. In all cases, the fingers already pressed on the right hand determine the pattern to be used. To add 3 more, use one of the following patterns:

*Case 1: (+3)*

When three fingers on the right hand are available for pressing, use them. *When three right-hand fingers are not available for pressing, use either:*

*Case 2: (5-2)*

When three right-hand fingers are not available but the fifth finger is available for pressing.

*Case 3: (10-7)*

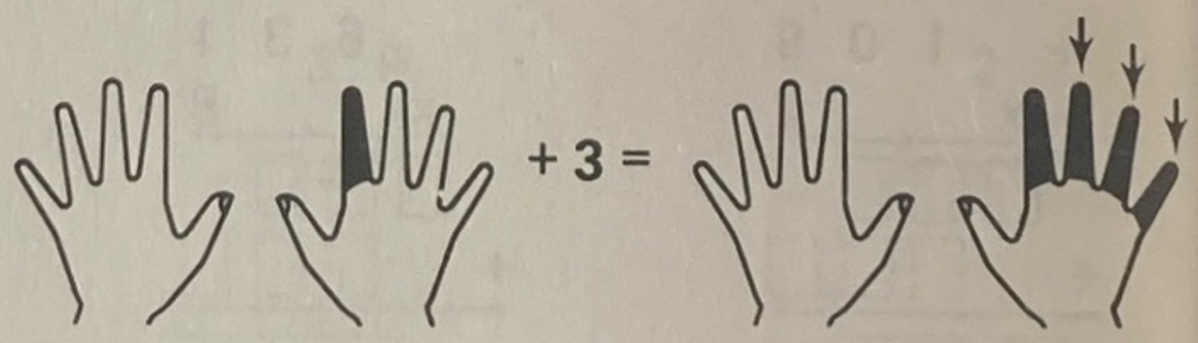
When neither three right-hand fingers nor the fifth finger are available for pressing.

**Note:** When clearing fingers, you must always work inward from the outside.

Illustration Example A: *Case 1: (+3)*

Three right-hand fingers are available for pressing. When we add 3 to the numbers 0, 1, 5 or 6 (already pressed), we must use the pattern that employs the three fingers still available for pressing.

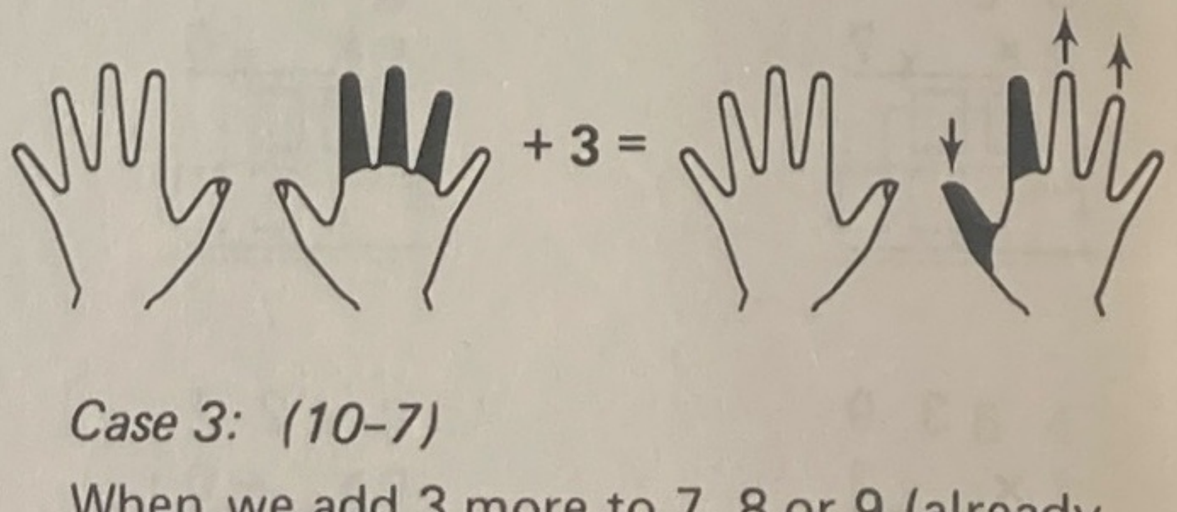
*1 is already pressed* (Press 3 directly)




*Case 2: (5-2)*

When we add 3 more to 2, 3, or 4 (already pressed), we must use the 5-2—press 5, clear 2—manipulation.

*3 is already pressed* (Press 5, Clear 2 simultaneously)




*Case 3: (10-7)*

When we add 3 more to 7, 8 or 9 (already pressed), we must use the pattern that employs the 10-7—press 10, clear 7—manipulation.

*8 is already pressed* (Press 10, Clear 7 simultaneously)

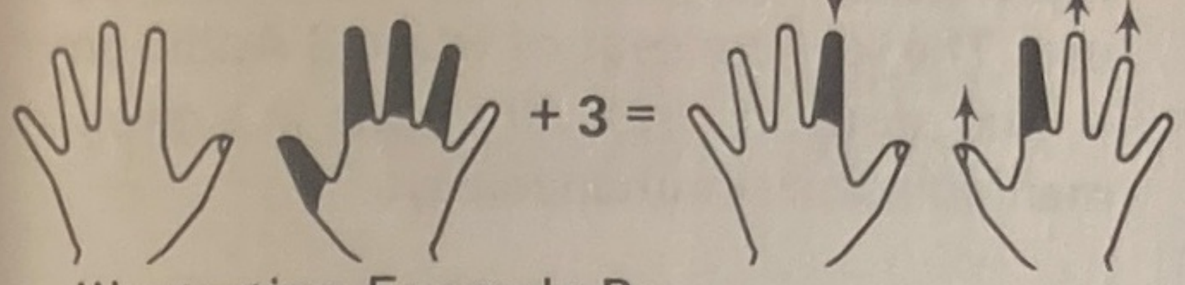


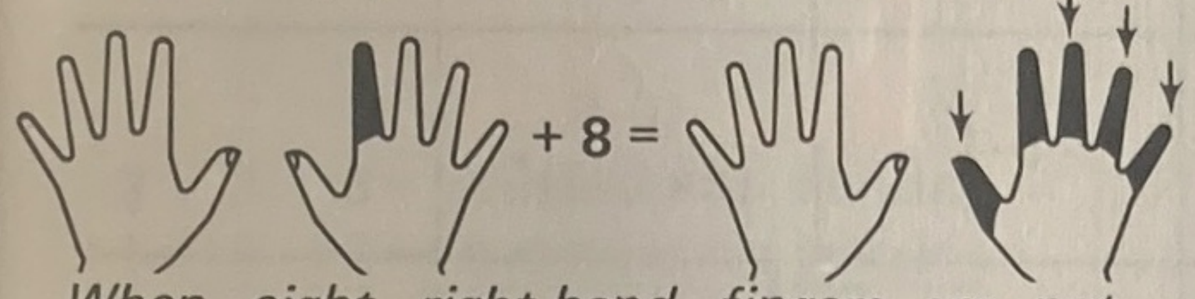

Illustration Example B:
(Adding the number 8)

There are three different ways of adding 8. Again it depends on which fingers are pressed on the right hand. To add 8, use one of the following patterns.

*Case 1: (+8)*

The number 8 is available for pressing. When we add 8 to the numbers 0 or 1 (already pressed), we must use the pattern that employs the fingers still available for pressing.

*1 is already pressed* (Press 8 directly)

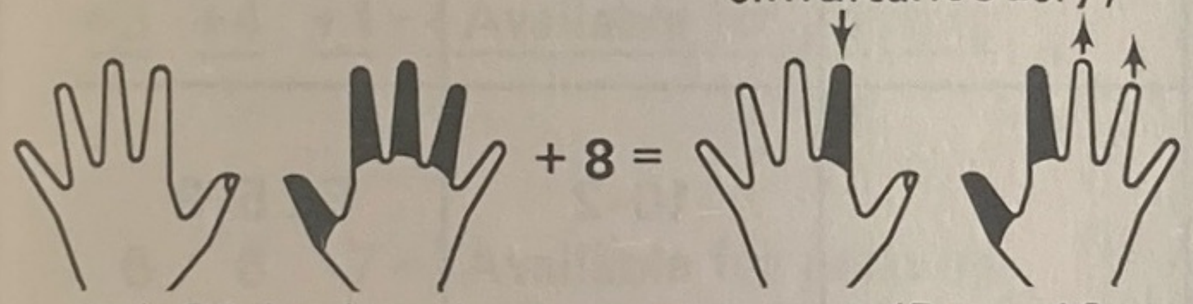



*When eight right-hand fingers are not available for pressing use either:*

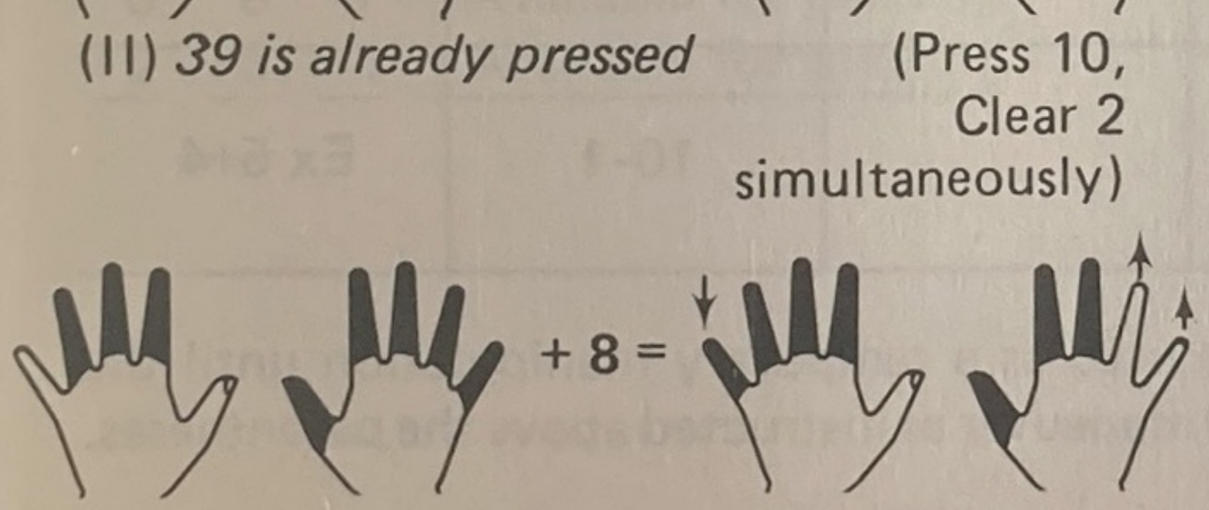
*Case 2: (10–2)*

When we add 8 more to the numbers 2, 3, 4, 7, 8, or 9 (already pressed), we must use the pattern that employs the 10–2–press 10, clear 2—manipulation.

(I) *8 is already pressed* (Press 10, Clear 2 simultaneously)



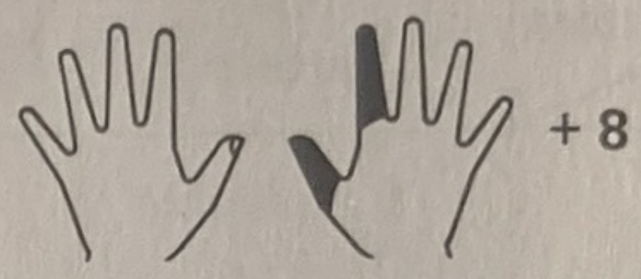

(II) *39 is already pressed* (Press 10, Clear 2 simultaneously)



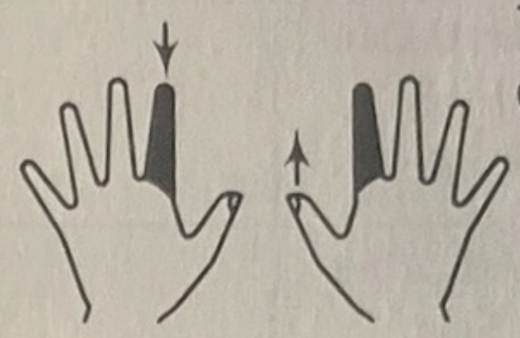

*Case 3: (Exchange 5+3)*

When we add 8 more to the numbers 5 or 6 (already pressed), we must use the pattern that employs the exchange 5 plus 3 manipulation.

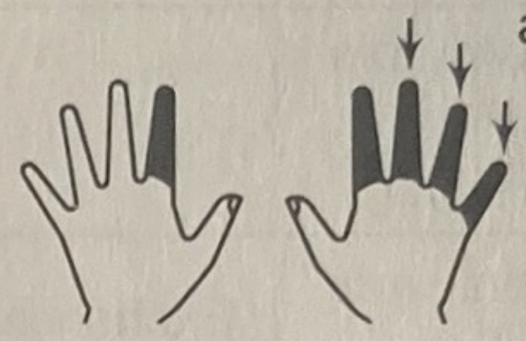
(I) *6 is already pressed*




A. Exchange 5 (10–5) Through press 10, clear 5 manipulation

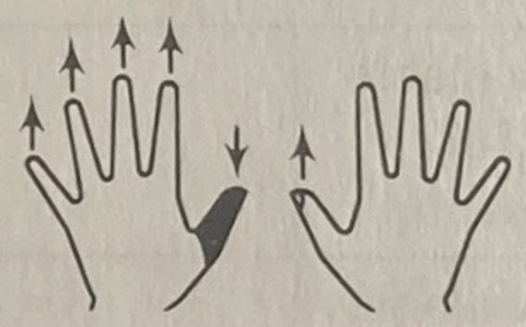


B. and simultaneously add 3 more

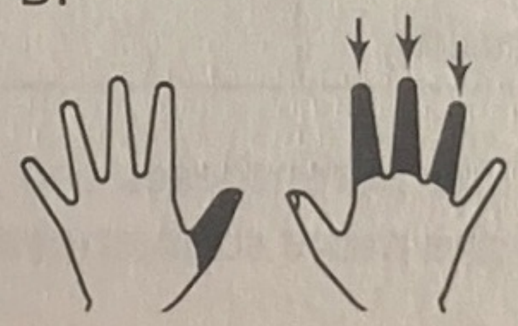


(II) *45 is already pressed*

A. Exchange 5 (10–5) Through the manipulation of Left Hand –Press 50 & Clear 40 and Right Hand– Clear 5



B. and simultaneously add 3 more



# ADDITION STAGE 3 PATTERN CHART

Addition Stage 3 manipulations are combinations of the following mathematical patterns employed in adding certain numbers. The fingers already pressed on the right hand determine which pattern to use. The key concept of Stage 3 Addition is that students press all fingers in a given manipulation simultaneously.

| Addend \ Pattern | Pattern 1 | Pattern 2 | Pattern 3 | Pattern 4 | Pattern 5 |
|---|---|---|---|---|---|
| 1 | 1 Available For Pressing | | 5-4 | 10-9 (Ex 5-4)* | |
| 2 | 2 Available For Pressing | | 5-3 | 10-8 (Ex 5-3)* | |
| 3 | 3 Available For Pressing | | 5-2 | 10-7 (Ex 5-2)* | |
| 4 | 4 Available For Pressing | | 5-1 | 10-6 (Ex 5-1)* | |
| 5 | 5 Available For Pressing | Exchange 5 (10-5) | | | |
| 6 | 6 Available For Pressing | | | 10-4 | Ex 5+1 |
| 7 | 7 Available For Pressing | | | 10-3 | Ex 5+2 |
| 8 | 8 Available For Pressing | | | 10-2 | Ex 5+3 |
| 9 | 9 Available For Pressing | | | 10-1 | Ex 5+4 |

*The instructions in the parentheses can be used as a temporary manipulation until the student gets used to the more sophisticated maneuver as instructed above the parentheses.

## EXERCISES

Pattern 1 & Pattern 1

1 1 1 ← Available for pressing
+1 +2 +3 ←+Available for pressing

1 1 1 ← Available for pressing
+5 +6 +7 ←+Available for pressing

1 2 2 ← Available for pressing
+8 +1 +2 ←+Available for pressing

2 2 2 ← Available for pressing
+5 +6 +7 ←+Available for pressing

3 3 3 ← Available for pressing
+1 +5 +6 ←+Available for pressing

4 5 5 ← Available for pressing
+5 +1 +2 ←+Available for pressing

5 5 6 ← Available for pressing
+3 +4 +1 ←+Available for pressing

6 6 7 ← Available for pressing
+2 +3 +1 ←+Available for pressing

Pattern 1 & Pattern 2

5 ← Available for pressing
+5 ←+Exchange 5—Press 10 Clear 5

6 ← Available for pressing
+5 ←+Exchange 5—Press 10 Clear 5

7 ← Available for pressing
+5 ←+Exchange 5—Press 10 Clear 5

8 ← Available for pressing
+5 ←+Exchange 5—Press 10 Clear 5

9 ← Available for pressing
+5 ←+Exchange 5—Press 10 Clear 5

Pattern 1 & Pattern 3

1 2 3 ← Available for pressing
+4 +4 +4 ←+Press 5 and Clear ___

4 2 3 ← Available for pressing
+4 +3 +3 ←+Press 5 and Clear ___

4 3 4 4 ← Available for pressing
+3 +2 +2 +1 ←+Press 5 and Clear ___

Pattern 1 & Pattern 4

1 2 3 ► Available for pressing
+9 +9 +9 ►+Press 10 and Clear ___

9 4 ► Available for pressing
+7 +6 ►+Press 10 and Clear ___

4 6 7 ► Available for pressing
+9 +9 +9 ►+Press 10 and Clear ___

9 6 ► Available for pressing
+6 +4 ►+Press 10 and Clear ___

8 9 2 ► Available for pressing
+9 +9 +8 ►+Press 10 and Clear ___

7 8 ► Available for pressing
+4 +4 ►+Press 10 and Clear ___

3 4 7 ► Available for pressing
+8 +8 +8 ►+Press 10 and Clear ___

9 7 ► Available for pressing
+4 +3 ►+Press 10 and Clear ___

8 9 3 ► Available for pressing
+8 +8 +7 ►+Press 10 and Clear ___

8 1 ► Available for pressing
+3 +9 ►+Press 10 and Clear ___

4 8 9 ► Available for pressing
+7 +7 +7 ►+Press 10 and Clear ___

9 8 ► Available for pressing
+3 +2 ►+Press 10 and Clear ___

4 9 8 ► Available for pressing
+6 +6 +7 ►+Press 10 and Clear ___

9 9 ► Available for pressing
+2 +1 ►+Press 10 and Clear ___

Pattern 1 & Pattern 5

5 6 7 ← Available for pressing
+6 +6 +6 ←+Exchange 5 and Press ___

8 5 6 ← Available for pressing
+6 +7 +7 ←+Exchange 5 and Press ___

7 5 6 ← Available for pressing
+7 +8 +8 ←+Exchange 5 and Press ___

5 6 5 ← Available for pressing
+9 +6 +9 ←+Exchange 5 and Press ___

7 5 8 ← Available for pressing
+7 +6 +6 ←+Exchange 5 and Press ___

Combination problems of Pattern 1 and Pattern 2.

| | | | |
|---|---|---|---|
| 1 (Available) | 8 (Available) | | |
| 2 (Available) | 5 (Exchange 5) | | |
| 5 (Available) | 5 (Available) | | |
| 5 (Exchange 5) | 1 (Available) | | |
| 1 (Available) | 5 (Exchange 5) | | |
| +5 (Available) | +5 (Available) | | |
| 5 | 7 | 3 | 5 |
| 5 | 5 | 5 | 5 |
| 5 | 7 | 5 | 6 |
| 4 | 5 | 5 | 5 |
| 5 | 5 | 1 | 6 |
| +5 | +5 | +5 | +5 |
| 5 | 2 | 5 | 1 |
| 3 | 6 | 5 | 5 |
| 5 | 5 | 9 | 3 |
| 5 | 5 | 5 | 5 |
| 5 | 5 | 5 | 5 |
| +1 | +6 | +5 | +5 |
| 1 (Available) | 7 (Available) | | |
| 5 (Available) | 5 (Exchange 5) | | |
| 1 (Available) | 5 (Available) | | |
| 5 (Exchange 5) | 5 (Exchange 5) | | |
| 6 (Available) | 7 (Available) | | |
| +5 (Exchange 5) | +5 (Exchange 5) | | |
| 6 (Available) | 5 (Available) | | |
| 5 (Exchange 5) | 2 (Available) | | |
| 6 (Available) | 1 (Available) | | |
| 5 (Exchange 5) | 5 (Exchange 5) | | |
| 6 (Available) | 5 (Available) | | |
| +5 (Exchange 5) | +1 (Available) | | |
| 5 (Available) | 6 (Available) | | |
| 5 (Exchange 5) | 1 (Available) | | |
| 1 (Available) | 5 (Exchange 5) | | |
| 8 (Available) | 6 (Available) | | |
| 5 (Exchange 5) | 5 (Exchange 5) | | |
| +5 (Available) | +1 (Available) | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 6 | 2 | 5 | 7 | 5 | 1 | 8 |
| 6 | 1 | 5 | 5 | 5 | 4 | 7 | 5 |
| 5 | 1 | 5 | 5 | 4 | 5 | 5 | 3 |
| 1 | 5 | 6 | 3 | 3 | 2 | 2 | 3 |
| 5 | 6 | 5 | 5 | 5 | 5 | 3 | 5 |
| +1 | +5 | +5 | +6 | +1 | +4 | +5 | +4 |

| | | | |
|---|---|---|---|
| 6 | 8 | 1 | 1 |
| 5 | 5 | 7 | 5 |
| 7 | 5 | 1 | 5 |
| 5 | 5 | 5 | 8 |
| 6 | 1 | 5 | 5 |
| +5 | +5 | +5 | +5 |

| | |
|---|---|
| 2 (Available) | 9 (Available) |
| 2 (Available) | 5 (Exchange 5) |
| 1 (5–4) | 3 (5–2) |
| 4 (Available) | 5 (Exchange 5) |
| 5 (Exchange 5) | 2 (Available) |
| +4 (5–1) | +3 (5–2) |

Combination problems of Patterns 1, 2, and 3.

| | |
|---|---|
| 2 (Available) | 3 (Available) |
| 7 (Available) | 5 (Available) |
| 5 (Exchange 5) | 5 (Exchange 5) |
| 4 (Press 5, Clear 1) | 3 (Press 5, Clear 2) |
| 5 (Exchange 5) | 5 (Exchange 5) |
| +4 (Press 5, Clear 1) | +4 (Press 5, Clear 1) |

| | |
|---|---|
| 9 (Available) | 2 (Available) |
| 5 (Exchange 5) | 2 (Available) |
| 2 (Press 5, Clear 3) | 1 (Press 5, Clear 4) |
| 5 (Exchange 5) | 3 (Available) |
| 3 (Available) | 5 (Exchange 5) |
| +1 (Press 5, Clear 4) | +2 (Press 5, Clear 3) |

| | | | |
|---|---|---|---|
| 2 (Available) | 4 (Available) | 4 | 6 | 4 | 4 |
| 6 (Available) | 1 (5–4) | 5 | 5 | 2 | 1 |
| 5 (Exchange 5) | 4 (Available) | 5 | 3 | 3 | 3 |
| 2 (5–3) | 5 (Exchange 5) | 3 | 2 | 5 | 1 |
| 3 (Available) | 1 (5–4) | 5 | 5 | 1 | 5 |
| +1 (Available) | +5 (Exchange 5) | +3 | +4 | +2 | +2 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 8 | 4 | 1 | 9 | 7 | 8 | 2 | 3 |
| 5 | 4 | 3 | 5 | 1 | 5 | 2 | 1 |
| 6 | 5 | 1 | 2 | 5 | 1 | 4 | 1 |
| 5 | 3 | 5 | 5 | 1 | 2 | 5 | 4 |
| 3 | 2 | 2 | 4 | 1 | 5 | 3 | 5 |
| +2 | +5 | +3 | +1 | +3 | +4 | +3 | +4 |

Combination problems of Patterns 1, 2, 3, and 4.

| | |
|---|---|
| 9 (Available) | 9 (Available) |
| 6 (Press 10, Clear 4) | 1 (Press 10, Clear 9) |
| 4 (Available) | 9 (Available) |
| 7 (Press 10, Clear 3) | 8 (Press 10, Clear 2) |
| 9 (Press 10, Clear 1) | 5 (Exchange 5) |
| +4 (Available) | +8 (Press 10, Clear 2) |

| | |
|---|---|
| 9 (Available) | 7 (Available) |
| 3 (Press 10, Clear 7) | 2 (Available) |
| 7 (Available) | 1 (Press 10, Clear 9) |
| 2 (Press 10, Clear 8) | 9 (Available) |
| 8 (Available) | 2 (Press 10, Clear 8) |
| +4 (Press 10, Clear 6) | +9 (Press 10, Clear 1) |

| | | | |
|---|---|---|---|
| 8 | 7 | 6 | 9 |
| 3 | 5 | 3 | 1 |
| 4 | 9 | 3 | 9 |
| 4 | 8 | 7 | 1 |
| 3 | 2 | 2 | 6 |
| +8 | +7 | +2 | +4 |

| | | | |
|---|---|---|---|
| 1 | 7 | 8 | 9 |
| 9 | 1 | 1 | 4 |
| 1 | 3 | 2 | 6 |
| 9 | 7 | 7 | 6 |
| 4 | 2 | 3 | 3 |
| +6 | +4 | +6 | +5 |

| | |
|---|---|
| 4 (Available) | 6 (Available) |
| 3 (5–2) | 4 (10–6) |
| 8 (10–2) | 6 (Available) |
| 3 (Available) | 4 (10–6) |
| 7 (10–3) | 6 (Available) |
| +5 (Exchange 5) | +4 (10–6) |

| | |
|---|---|
| 8 (Available) | 6 (Available) |
| 2 (10–8) | 3 (Available) |
| 8 (Available) | 3 (10–7) |
| 2 (10–8) | 7 (Available) |
| 8 (Available) | 6 (10–4) |
| +3 (10–7) | +4 (Available) |

| | | | |
|---|---|---|---|
| 2 | 3 | 5 | 1 |
| 6 | 1 | 1 | 3 |
| 9 | 9 | 2 | 9 |
| 9 | 5 | 1 | 3 |
| 4 | 3 | 2 | 3 |
| +8 | +9 | +4 | +3 |

| | | | |
|---|---|---|---|
| 2 | 9 | 1 | 2 |
| 2 | 1 | 3 | 2 |
| 8 | 1 | 9 | 2 |
| 8 | 7 | 5 | 5 |
| 8 | 3 | 9 | 8 |
| +2 | +4 | +1 | +3 |

Combination problems of Patterns 1, 2, 3, 4, and 5

| | |
|---|---|
| 5 (Available) | 3 (Available) |
| 9 (Ex. 5, Press 4) | 3 (Press 5, Clear 2) |
| 2 (Press 5, Clear 3) | 7 (Ex. 5, Press 2) |
| 8 (Ex. 5, Press 3) | 5 (Available) |
| 7 (Press 10, Clear 3) | 6 (Ex. 5, Press 1) |
| +3 (Available) | +1 (Press 5, Clear 4) |

| | |
|---|---|
| 7 (Available) | 8 (Available) |
| 7 (Ex. 5, Press 2) | 1 (Available) |
| 5 (Available) | 6 (Press 10, Clear 4) |
| 6 (Press 10, Clear 4) | 8 (Ex. 5, Press 3) |
| 6 (Ex. 5, Press 1) | 3 (Press 5, Clear 2) |
| +4 (Press 5, Clear 1) | +9 (Press 10, Clear 1) |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 4 | 1 | 6 | 2 | 4 | 9 | 4 | 8 |
| 3 | 5 | 7 | 5 | 6 | 1 | 7 | 3 |
| 9 | 9 | 5 | 7 | 8 | 6 | 7 | 4 |
| 8 | 9 | 6 | 2 | 2 | 4 | 6 | 8 |
| 2 | 4 | 1 | 7 | 7 | 6 | 8 | 3 |
| +7 | +6 | +8 | +9 | +7 | +8 | +6 | +7 |

| | | | |
|---|---|---|---|
| 4 | 1 | 3 | 6 |
| 2 | 7 | 3 | 5 |
| 8 | 6 | 6 | 4 |
| 5 | 1 | 4 | 9 |
| 6 | 7 | 8 | 2 |
| +9 | +4 | +5 | +6 |

## EXERCISES

Transfer Carry-over Number Directly From Left Hand to Right Hand.

| | |
|---|---|
| 5,421 | 3,107 |
| 8,692 | 5,865 |
| 8,527 | 4,296 |
| +4,096 | +6,417 |

| | |
|---|---|
| 8,019 | 9,506 |
| 742 | 898 |
| 58 | 3,451 |
| +6,135 | + 627 |

| | |
|---|---|
| 6,926 | 581 |
| 4,073 | 4,509 |
| 18 | 526 |
| 503 | 9,978 |
| 4,296 | 50 |
| + 761 | +6,421 |

| | |
|---|---|
| 3,472 | 6,961 |
| 6,015 | 4,258 |
| 9,381 | 3,742 |
| +7,635 | +8,109 |

2 (Available)
4 (5–1)
7 (Ex. 5+2)
5 (Available)
6 (Ex. 5+1)
+4 (5–1)

6 (Available)
7 (Ex. 5+2)
3 (5–2)
8 (Ex. 5+3)
9 (10–1)
+3 (5–2)

9 (Available)
6 (10–4)
9 (Ex. 5+4)
2 (5–3)
7 (Ex. 5+2)
+3 (5–2)

6 (Available)
7 (Ex. 5+2)
2 (5–3)
9 (Ex. 5+4)
1 (5–4)
+9 (Ex. 5+4)

| | | | |
|---|---|---|---|
| 6 | 9 | 8 | 6 |
| 9 | 6 | 5 | 8 |
| 7 | 7 | 7 | 6 |
| 4 | 5 | 6 | 5 |
| 7 | 7 | 6 | 9 |
| +5 | +2 | +7 | +1 |

| | | | |
|---|---|---|---|
| 741 | 352 | 594 | 9,607 |
| 2,567 | 16 | 276 | 2,184 |
| 306 | 4,970 | 30 | 593 |
| + 89 | + 853 | 6,017 | 162 |
| | | 8,539 | 55 |
| | | + 81 | + 90 |
| 67 | 1,549 | | |
| 58 | 26 | 4,921 | 6,761 |
| 309 | 308 | 9,683 | 4,289 |
| 4,596 | 6,937 | 2,940 | 3,906 |
| 9,781 | 510 | +3,575 | +5,487 |
| + 302 | + 78 | | |
| | | 69 | 7,034 |
| 4,761 | 8,437 | 520 | 275 |
| 8,028 | 9,576 | 741 | 861 |
| 4,689 | 3,124 | +9,830 | + 93 |
| +5,312 | +2,069 | | |
| | | 151 | 71 |
| | | 2,370 | 69 |
| 1,524 | 751 | 869 | 5,024 |
| 363 | 63 | 25 | 810 |
| 17 | 5,908 | 6,804 | 783 |
| +9,805 | +6,429 | + 46 | +9,152 |

# MULTIPLICATION STAGE 3

Stage 3 Multiplication is divided into two sections. In the first section, we discuss conventional multiplication of multi-digit multipliers. Within this section, we discuss a slight change in the format that teachers should use when first teaching this subject. We go on to show how the carry-over numbers can be transferred directly and thus the speed of calculation can be improved. In the second section we discuss a new format for multiplying multi-digit numbers. With the new format, we show how you can continue to separate the two operations of multiplication and carry-over and thus improve your speed. Since the format is new, it is presented first in a preliminary format and then a final format. This method we call HYPAI PYRAMID Multiplication.

Before proceeding to third stage multiplication, we must issue one word of caution. We presume, in presenting third stage multiplication, that the multiplication facts are memorized. If the students do not know their times tables, they should remain at Stage 2 Multiplication. Only after they have memorized the tables, can they benefit from Stage 3.

## Conventional Multiplication

Conventional multiplication attempts to make explicit to the students what is occurring when they start to multiply with multi-digit numbers. They usually explain the method by presenting the problems in a format such as the following.

| | | | | |
|---|---|---|---|---|
| | | 9 | 7 | 3 |
| | x | | | 5 |
| Thousands | Hundreds | Tens | Units | |
| | | 1 | 5 | ◄ 5 x 3 |
| | 3 | 5 | 0 | ◄ 5 x 70 |
| +4 | 5 | 0 | 0 | ◄ 5 x 900 |
| 4 | 8 | 6 | 5 | ◄ 5 x 973 |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | 6 | 4 | 2 | |
| | | x | 3 | 7 | 6 | |
| Hundred Thousands | Ten Thousands | Thousands | Hundreds | Tens | Units | |
| | | 3 | 8 | 5 | 2 | ◄ 6 x 642 |
| | 4 | 4 | 9 | 4 | 0 | ◄ 70 x 642 |
| +1 | 9 | 2 | 6 | 0 | 0 | ◄300 x 642 |
| 2 | 4 | 1 | 3 | 9 | 2 | ◄376 x 642 |

Although this method is an excellent approach to teaching the proper places when multiplying multi-digit factors, the writing of the zeroes is required in order to preserve the proper places of the partial products.

The place is found by first writing down all the implicit zeroes and then moving from right to left once the first explicit digit of the product is found. Thus in multiplying 70 x 642, the students are instructed to consider this as (10 x 7) x 642 and are told to write the 0 for the 10 down immediately and then proceed as if they were multiplying 7 x 642. Moreover, heavy emphasis is placed on identifying the column verbally.

Although this method serves the students well, they sometimes run into trouble when they are shown how to accelerate their calculating speed by deleting the place holding zeroes. Moreover, some slow students have difficulty when this method is first introduced. Some of these difficulties can be alleviated by including a visual aid.

We ask you to examine the HYPAI Place Holding Chart. The first thing you will notice is that we have inserted small zeroes into the multipliers. The number 921 was rewritten as 9oo2o1. This is a short hand notation for concentrating on the fact that the number could have been rewritten as 900 + 20 + 1. In a similar manner, we have rewritten the other multiplier as 4oo3o2.

The second feature of the chart is that above the columns where the partial products are written, we have displayed what we call a verbal anchor and a visual anchor. For instance, above the 3rd column from the right, the verbal anchor is hundreds. The students must be made to understand that any digit in that column is expressing a set of hundreds. Below the word "hundreds" you will find two small zeroes in the row that contains the visual anchors. The students can now visualize any number in this column by saying it is followed by two zeroes, at least implicitly. The final feature of the chart is that it points out the precise place for each and every partial product. Starting from this chart, and using the instructions of how to deal with carry-over numbers, which follow immediately, we believe you will improve your conventional instruction.

# HYPAI VERTICAL PLACE HOLDING CHART*

## (Conventional Way—Multiplication Stage 3)

900 20 1
x 400 30 2

| | Hundred Thousands | Ten Thousands | Thousands | Hundreds | Tens | Units | Verbal Anchor | |
|---|---|---|---|---|---|---|---|---|
| | | 000 0 | 000 | 00 | 0 | | Visual Anchor | |
| (A) Partial Product | | | 1 | 8 | 4 | 2 | | (A) |
| (B) Partial Product | | 2 | 7 | 6 | 3 | | | (B) |
| (C) Partial Product | 3 | 6 | 8 | 4 | | | | (C) |

(No zero implied) 2 x 1
(One zero implied) 2 x 20
(Two zeroes implied) 2 x 900
900 20 1 x 2

(One zero implied) 30 x 1
(Two zeroes implied) 30 x 20
(Three zeroes implied) 30 x 900
900 20 1 x 30

(Two zeroes implied) 400 x 1
(Three zeroes implied) 400 x 20
(Four zeroes implied) 400 x 900
900 20 1 x 400

*From this point on, we will be using the conventional notation for multiplication vertically. In stages 1 & 2, the lower number was the number being repeatedly added. Since we presume that the tables are memorized, we switch back to conventional notation.

Utilizing the HYPAI Place Holding Chart and Hand Transfers

(EXAMPLE I)

5o 9
x 6

| Hundreds | Tens | Units | Verbal Anchor |
|---|---|---|---|
| | o | | Visual Anchor |
| 3 | 5 | 4 | 5o9 |
| | 6x5o | 6x9 | x 6 |

The key to utilizing the Place Holding Chart is the question: "What column does the right hand belong to?" Consider the example above. The first operation is to multiply 6 x 9. As mentioned at the beginning of this chapter, we presume that the students know this answer is 54. They must then ask themselves are there any implicit zeroes involved, i.e. "Are there any zeroes in the factors?" Since they were multiplying a unit times a unit, there are no implicit zeroes. They should write the 4 directly into the units' column. They should be able to reply that the 4 is a unit and that the units' column has no implicit zeroes.

## Hand Transfer

After they have written the 4 from their first partial product, they should immediately transfer the carry-over 5 onto their *right* hand. They should be made to understand that the right hand will now be counting tens and therefore any number that is registered on that hand is implicitly followed by a zero. To the 5 on the right hand, they should add the product of 6 x 5o. Once again, we presume they know the product and now their hands should be registering 35. The question is where to place this number. The answer may be found visually by reverting to the fact that the right hand was counting numbers with one implicit zero. The answer is found verbally by asking, "How many tens are on the right hand?" Since the number on the right hand must be followed by a zero, the right hand number should be placed in the column with the one zero at the top, the tens' column. Since you have now finished all the required multiplying, you can place the carry-over number on the left hand into the next column, the hundreds' column. The importance of using the HYPAI Place Holding Chart becomes clearer when the digits in the multiplier are expanded.

## Discarding the Hand Transfer

The transfer is suggested for those students, who although they know their tables, are slightly insecure with them. If they become completely confident of their tables and are able to mentally calculate a product and at the same time keep track of a carry-over number, there is no need to use the transfer. The transfer is used only by those students who are still confused by the dual operation of addition and multiplication.

(EXAMPLE II)

5o 8
x 4o 3

| Thousands | Hundreds | Tens | Units | Verbal Anchor |
|---|---|---|---|---|
| | oo | o | | Visual Anchor |
| | 1 | 7 | 4 | 5o8 |
| | | 3x5o | 3x8 | x 3 |
| 2 | 3 | 2 | | 5o8 |
| + | | 4o x 5o | 4o x 8 | x4o |

As the digits in the multipliers expand, the value of the HYPAI Place Holding Chart becomes even more apparent. In the above example the first task is to multiply 3 x 8. Since there are no implicit zeroes in either of the multipliers, the right hand value of this product, 4, can be written down immediately in the units' column and the value of the carry-over, 2, transferred to the right hand. To this must be added the product of 3 x 5o. This is done by pressing another 15 on the fingers. The right hand should now have a 7 pressed and the left hand should have 10 pressed. Since the right hand shows a 7, the question becomes which column should the 7 be entered into. We can answer that question in two ways. (1) Because there was an implicit zero in one of the multipliers, 5o, the 7 of the right hand must be placed into the column headed with one zero; or (2) after we wrote down the 4 into the units' column, the right hand started counting tens and therefore the entry must be made in the tens' column. Once you have decided where to place the 7 from the right hand, the number on the left hand is easy to place. Since it is a carry-over number, it goes into the next left hand column, the hundreds' column.

As in the conventional approach, we continue on with the problem using a new line. The first multiplication to be performed is 4o x 8. Once again, we treat it as if it were 4 x 8, but take care to recognize that there will be one implicit zero in the product. Since the answer is 32, we know that the right hand answer is 2. The question becomes where should the 2 be placed? We find the answer by counting the implicit zeroes in the multipliers. Since there was one implicit zero in 4o, the 2 should be placed in the column with one zero. After you have written down the 2, you should transfer the 3 from the carry-over onto the right hand. To this you must add the product of 4o x 5o. You will actually be pressing 20, and your hands should end up reading 23. Again, you ask yourself which column the 3 on the right hand belongs to. Since there was one implicit zero in *each* multiplier, the 3 must be placed into the column with *two* zeroes, the hundreds' column. The carry-over number on the left hand, 2, is placed immediately into the next column, the thousands' column. Note how, by using this chart, you can justify the placement of every digit, not just the first digit in the multi-digit product. You complete your answer by adding the two lines.

(EXAMPLE III)

6oo 4o 2
x 3oo 7o 6

| Hundred Thousands | Ten Thousands | Thousands | Hundreds | Tens | Units | Verbal Anchor |
|---|---|---|---|---|---|---|
| | ooo<br>o | ooo | oo | o | | Visual Anchor |
| | | 3 | 8 | 5 | 2 | 6oo4o2<br>x 6 |
| | | | 6x4oo | 6x4o | 6x2 | |
| | 4 | 4 | 9 | 4 | | 6oo4o2<br>x 7o |
| | | 7o x 6oo | 7o x 4o | 7o x 2 | | |
| 1 | 9 | 2 | 6 | | | 6oo4o2<br>x3oo |
| + | 3oo x 6oo | 3oo x 4o | 3oo x 2 | | | |

We show one more example using 3-digit multipliers. The first step in the above problem is to multiply 6 x 2. Since there is no implicit zero in either of the multipliers, the right hand number, 2, is written immediately into the units' column, and the left hand number, 1, is transferred onto the right hand. To this must be added the product of 6 x 4o. Having pressed a 24, your fingers should read 25. You ask yourself where to place the 5 on the right hand and you answer, "Into the column with one zero" because one of the factors had an implicit zero. A 5 should therefore be placed in the tens' column and the 2 on the left hand transferred to the right hand. To this new carry-over number you must add the product of 6 x 6oo. You therefore press another 36 and wind up with the fingers reading 38. To find out where the 8 should be written down, you check the factors and find that one of them contained two implicit zeroes so you write it into the column with two zeroes. The carry-over, 3, on your left hand, is written down one column to the left.

You now approach the second line and multiply 7o x 2. The answer is 14. You then ask yourself, "into which column should the 4 be placed?" Since one of the factors had one implicit zero you write the 4 into the column headed with one zero. You then transfer the 1 to the right hand. To this must be added the product of 7o x 4o. Once you have added 28, your hands read 29. You know that the 9 must be placed into the column with two zeroes because each of the factors had one implicit zero. After you have written the 9 into the hundreds' column, you transfer the 2 from your left hand to your right and add the product of 7o x 6oo. After you have added the 42, your hands should read 44 and you check where to place the number on the right hand. Since the factors had three implicit zeroes, the right hand number must go into the column with three zeroes. Since the left hand is a carry-over number, it is placed into the next column.

You now calculate the third line and the first multiplication is 3oo x 2. You know the answer is 6 and it goes into the column with two zeroes because of its factors had two implicit zeroes. There is no carry-over number. The next calculation is 3oo x 4o. You know the answer to this to be 12, and

you know that the 2 should be placed into the column with three zeroes since there were three implicit zeroes in the factors. You transfer the 1 from the left to the right hand and add the product of 3oo x 6oo. Your hands should now read 19 and by now you should realize the 9 goes into the column with four zeroes. The carry-over of 1 is placed into the next column. You complete the problem by adding all three lines.

## Retrieval of Carry-overs

If the carry-over number is forgotten, it can easily be retrieved. Just go back to the *prior column* and perform the necessary multiplication. Then continue to press and count until the right hand shows the number written in the product. The number you count is the carry-over number belonging to the prior right hand column. The number on your left hand is the number that you originally lost, and had to retrieve.

## Summary

The HYPAI Vertical Place Holding Chart was developed to assist the students in recognizing which place to put partial products. It improves upon the conventional since it has a means of showing a place for each partial product, not just the first one in a row. It also assists the teaching of place value by showing that any multi-digit number contains many implicit digits e.g. 53,712 = 50,000 + 3,000 + 700 + 10 + 2.

## HYPAI VERTICAL PLACE HOLDING METHOD

| | Ten Thousands | Thousands | Hundreds | Tens | Units | Verbal Anchor |
|---|---|---|---|---|---|---|
| | 0000 | 000 | 00 | 0 | | Visual Anchor |
| | 5 | 9 | 8 | 6 | 4 | |
| 1 x 4 | (No zero implied) | | | | 4<br>4 | 1 x 4 |
| 3o x 2 | (One zero implied) | | | 6<br>6 | 0 | 30 x 2 |
| 2o x 4o | (Two zeroes implied) | | 8<br>8 | 0 | 0 | 20 x 40 |
| 3oo x 3o | (Three zeroes implied) | 9<br>9 | 0 | 0 | 0 | 300 x 30 |
| 5oo x 1oo | (Four zeroes implied) 5<br>5 | 0 | 0 | 0 | 0 | 500 x 100 |

## CONVENTIONAL WAY
## (Multi-Digit x Single Digit)

Transfer Carry-over Directly From Left Hand to Right Hand

## EXERCISES

| | | |
|---|---|---|
| | 5o | 9 |
| | x | 6 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |

| | | |
|---|---|---|
| | 9o | 3 |
| | x | 6 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |

| | | |
|---|---|---|
| | 8o | 3 |
| | x | 7 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |

| | | |
|---|---|---|
| | 2o | 6 |
| | x | 1 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |

| | | |
|---|---|---|
| | 1o | 8 |
| | x | 5 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |

| | | |
|---|---|---|
| | 1o | 5 |
| | x | 3 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |

| | | |
|---|---|---|
| | 4o | 1 |
| | x | 9 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |

| | | |
|---|---|---|
| | 6o | 2 |
| | x | 9 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |

| | | |
|---|---|---|
| | 4o | 0 |
| | x | 8 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |

| | | |
|---|---|---|
| | 3o | 7 |
| | x | 0 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |

| | 7o | 2 |
|---|---|---|
| x | | 2 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |
| | | |

| | 6o | 0 |
|---|---|---|
| x | | 7 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |
| | | |

| | 7o | 0 |
|---|---|---|
| x | | 4 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |
| | | |

| | 3o | 4 |
|---|---|---|
| x | | 1 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |
| | | |

| | 2o | 7 |
|---|---|---|
| x | | 5 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |
| | | |

| | 6o | 3 |
|---|---|---|
| x | | 3 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |
| | | |

| | 8o | 5 |
|---|---|---|
| x | | 8 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |
| | | |

| | 7o | 8 |
|---|---|---|
| x | | 4 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |
| | | |

| | 9o | 1 |
|---|---|---|
| x | | 2 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |
| | | |

| | 5o | 9 |
|---|---|---|
| x | | 0 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |
| | | |

| | 1o | 2 |
|---|---|---|
| x | | 2 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |

| | 4o | 0 |
|---|---|---|
| x | | 8 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |

| | 9o | 7 |
|---|---|---|
| x | | 3 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |

| | 3o | 5 |
|---|---|---|
| x | | 7 |

| Hundreds | Tens | Units |
|---|---|---|
| | 0 | |
| | | |

| | 4o | 6 |
|---|---|---|
| x | 7o | 8 |

| Thousands | Hundreds | Tens | Units |
|---|---|---|---|
| | 00 | 0 | |
| | | | |
| + | | | |
| | | | |

| | 9o | 4 |
|---|---|---|
| x | 6o | 5 |

| Thousands | Hundreds | Tens | Units |
|---|---|---|---|
| | 00 | 0 | |
| | | | |
| + | | | |
| | | | |

| | 6o | 7 |
|---|---|---|
| x | 2o | 1 |

| Thousands | Hundreds | Tens | Units |
|---|---|---|---|
| | 00 | 0 | |
| | | | |
| + | | | |
| | | | |

## EXERCISES
## (Multi-digits x Multi-digits)

| | 3o | 5 |
|---|---|---|
| x | 4o | 7 |

| Thousands | Hundreds | Tens | Units |
|---|---|---|---|
| | 00 | 0 | |
| | | | |
| + | | | |
| | | | |

| | 6o | 9 |
|---|---|---|
| x | 8o | 7 |

| Thousands | Hundreds | Tens | Units |
|---|---|---|---|
| | 00 | 0 | |
| | | | |
| + | | | |
| | | | |

| | | | |
|---|---|---|---|
| | | 70 | 3 |
| | | x 40 | 2 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | | |
|---|---|---|---|
| | | 30 | 7 |
| | | x 60 | 4 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | | |
|---|---|---|---|
| | | 70 | 0 |
| | | x 90 | 4 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | | |
|---|---|---|---|
| | | 60 | 4 |
| | | x 40 | 5 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | | |
|---|---|---|---|
| | | 80 | 7 |
| | | x 50 | 3 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | | |
|---|---|---|---|
| | | 10 | 3 |
| | | x 80 | 4 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | | |
|---|---|---|---|
| | | 60 | 3 |
| | | x 20 | 7 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | | |
|---|---|---|---|
| | | 20 | 6 |
| | | x 40 | 1 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | 7o | 3 |
|---|---|---|---|
| | | x 5o | 9 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | 4o | 8 |
|---|---|---|---|
| | | x 2o | 7 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | 5o | 7 |
|---|---|---|---|
| | | x 4o | 6 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | 5o | 4 |
|---|---|---|---|
| | | x 1o | 3 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | 6o | 1 |
|---|---|---|---|
| | | x 3o | 8 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | 3o | 0 |
|---|---|---|---|
| | | x 6o | 1 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | 9o | 2 |
|---|---|---|---|
| | | x 3o | 6 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | 5o | 0 |
|---|---|---|---|
| | | x 3o | 4 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | 9o | 8 |
|---|---|---|---|
| | | x 6o | 7 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | 1oo | 7o | 4 |
|---|---|---|---|---|
| | | x | 8o | 3 |
| Ten Thousands | Thousands | Hundreds | Tens | Units |
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | 3o | 2 |
|---|---|---|---|
| | | x 3o | 1 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | 9oo | 2o | 5 |
|---|---|---|---|---|
| | | x | 1o | 4 |
| Ten Thousands | Thousands | Hundreds | Tens | Units |
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | 1o | 5 |
|---|---|---|---|
| | | x 5o | 2 |
| Thousands | Hundreds | Tens | Units |
| | 00 | 0 | |
| | | | |
| + | | | |

| | | 6oo | 7o | 3 |
|---|---|---|---|---|
| | | x | 8o | 4 |
| Ten Thousands | Thousands | Hundreds | Tens | Units |
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | 3oo | 5o | 6 |
|---|---|---|---|---|
| | | x | 9o | 2 |
| Ten Thousands | Thousands | Hundreds | Tens | Units |
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | 9oo | 2o | 5 |
|---|---|---|---|---|
| | | x | 1o | 6 |
| Ten Thousands | Thousands | Hundreds | Tens | Units |
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | 300 | 70 | 8 |
| | x | 40 | 2 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | 200 | 90 | 8 |
| | x | 40 | 7 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | 200 | 90 | 1 |
| | x | 70 | 5 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | 600 | 90 | 8 |
| | x | 30 | 4 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | 800 | 30 | 7 |
| | x | 20 | 6 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | 900 | 00 | 7 |
| | x | 50 | 6 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | 500 | 60 | 4 |
| | x | 30 | 4 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | 600 | 10 | 2 |
| | x | 80 | 3 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | 100 | 40 | 6 |
| | x | 50 | 0 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | 600 | 70 | 0 |
| | x | 10 | 7 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | 700 | 30 | 6 |
| | x | 80 | 9 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | | 40 | 7 |
| | x 300 | 60 | 8 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | 800 | 20 | 6 |
| | x | 90 | 9 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | | 20 | 1 |
| | x 900 | 00 | 5 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | 500 | 80 | 1 |
| | x | 30 | 4 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | | 80 | 3 |
| | x 700 | 50 | 2 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| + | | | | |

| | | | | | |
|---|---|---|---|---|---|
| | | | | 1o | 6 |
| | | x | 7oo | 4o | 0 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| | | | | |
| + | | | | |

| | | | | | |
|---|---|---|---|---|---|
| | | | | 3o | 7 |
| | | x | 5oo | 0o | 6 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| | | | | |
| + | | | | |

| | | | | | |
|---|---|---|---|---|---|
| | | | | 5o | 0 |
| | | x | 1oo | 6o | 3 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| | | | | |
| + | | | | |

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|---|---|---|---|---|---|
| | | | | 6o | 0 |
| | | x | 7oo | 2o | 1 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| | | | | |
| + | | | | |

| | | | | | |
|---|---|---|---|---|---|
| | | | | 2o | 4 |
| | | x | 8oo | 1o | 5 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| | | | | |
| + | | | | |

| | | | | | |
|---|---|---|---|---|---|
| | | | | 6o | 3 |
| | | x | 7oo | 9o | 8 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| | | | | |
| + | | | | |

| | | | | | |
|---|---|---|---|---|---|
| | | | | 4o | 9 |
| | | x | 2oo | 6o | 1 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| | | | | |
| + | | | | |

| | | | | | |
|---|---|---|---|---|---|
| | | | | 7o | 0 |
| | | x | 4oo | 2o | 5 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | | | 1o | 7 |
| | | x 7oo | 9o | 4 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | | | 2o | 5 |
| | | x 1oo | 7o | 4 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | | | 9o | 1 |
| | | x 4oo | 9o | 3 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | | | 8o | 2 |
| | | x 6oo | 1o | 0 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | | | 3o | 9 |
| | | x 5oc | 2o | 6 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| | | | | |
| + | | | | |

| | | | | |
|---|---|---|---|---|
| | | | 5o | 3 |
| | | x 2oo | 6o | 3 |

| Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|
| | 000 | 00 | 0 | |
| | | | | |
| | | | | |
| + | | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | 3oo | 4o | 5 |
| | | x | 6oo | 9o | 1 |

| Hundred Thousands | Ten Thousands | Thou-sands | Hun-dreds | Tens | Units |
|---|---|---|---|---|---|
| | 000<br>0 | 000 | 00 | 0 | |
| | | | | | |
| | | | | | |
| | | | | | |
| + | | | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | 6oo | 4o | 1 |
| | | x | 3oo | 8o | 7 |

| Hundred Thousands | Ten Thousands | Thou-sands | Hun-dreds | Tens | Units |
|---|---|---|---|---|---|
| | 000<br>0 | 000 | 00 | 0 | |
| | | | | | |
| | | | | | |
| | | | | | |
| + | | | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | 8oo | 6o | 9 |
| | | x | 4oo | 3o | 2 |

| Hundred Thousands | Ten Thousands | Thou-sands | Hun-dreds | Tens | Units |
|---|---|---|---|---|---|
| | 000<br>0 | 000 | 00 | 0 | |
| | | | | | |
| | | | | | |
| | | | | | |
| + | | | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | 3oo | 2o | 1 |
| | | x | 7oo | 4o | 0 |

| Hundred Thousands | Ten Thousands | Thou-sands | Hun-dreds | Tens | Units |
|---|---|---|---|---|---|
| | 000<br>0 | 000 | 00 | 0 | |
| | | | | | |
| | | | | | |
| | | | | | |
| + | | | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | 1oo | 7o | 3 |
| | | x | 9oo | 0o | 5 |

| Hundred Thousands | Ten Thousands | Thou-sands | Hun-dreds | Tens | Units |
|---|---|---|---|---|---|
| | 000<br>0 | 000 | 00 | 0 | |
| | | | | | |
| | | | | | |
| | | | | | |
| + | | | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | 2oo | 0o | 8 |
| | | x | 1oo | 3o | 5 |

| Hundred Thousands | Ten Thousands | Thou-sands | Hun-dreds | Tens | Units |
|---|---|---|---|---|---|
| | 000<br>0 | 000 | 00 | 0 | |
| | | | | | |
| | | | | | |
| | | | | | |
| + | | | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | 5oo | 2o | 9 |
| | | x | 4oo | 3o | 7 |

| Hundred Thousands | Ten Thousands | Thou-sands | Hun-dreds | Tens | Units |
|---|---|---|---|---|---|
| | 000<br>0 | 000 | 00 | 0 | |
| | | | | | |
| | | | | | |
| | | | | | |
| + | | | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | 8oo | 0o | 3 |
| | | x | 5oo | 7o | 3 |

| Hundred Thousands | Ten Thousands | Thou-sands | Hun-dreds | Tens | Units |
|---|---|---|---|---|---|
| | 000<br>0 | 000 | 00 | 0 | |
| | | | | | |
| | | | | | |
| | | | | | |
| + | | | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | 1oo | 8o | 6 |
| | | x | 2oo | 5o | 0 |

| Hundred Thousands | Ten Thousands | Thou-sands | Hun-dreds | Tens | Units |
|---|---|---|---|---|---|
| | 000<br>0 | 000 | 00 | 0 | |
| | | | | | |
| | | | | | |
| | | | | | |
| + | | | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | 4oo | 8o | 9 |
| | | x | 2oo | 6o | 1 |

| Hundred Thousands | Ten Thousands | Thou-sands | Hun-dreds | Tens | Units |
|---|---|---|---|---|---|
| | 000<br>0 | 000 | 00 | 0 | |
| | | | | | |
| | | | | | |
| | | | | | |
| + | | | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | 9oo | 7o | 0 |
| | | x | 6oo | 4o | 1 |

| Hundred Thousands | Ten Thousands | Thou-sands | Hun-dreds | Tens | Units |
|---|---|---|---|---|---|
| | 000<br>0 | 000 | 00 | 0 | |
| | | | | | |
| | | | | | |
| | | | | | |
| + | | | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | 3oo | 5o | 2 |
| | | x | 1oo | 0o | 7 |

| Hundred Thousands | Ten Thousands | Thou-sands | Hun-dreds | Tens | Units |
|---|---|---|---|---|---|
| | 000<br>0 | 000 | 00 | 0 | |
| | | | | | |
| | | | | | |
| | | | | | |
| + | | | | | |

## HYPAI Pyramid Multiplication—Stage 3 Format A

In Stage 2 Multiplication, we presented a new format that you could use when multiplying a multi-digit number by a single digit. The purpose of that format was to separate the problem into two parts. First you performed all of the multiplications, and then did the required adding. The students did not have to deal with carry-over numbers under that format.

Using the HYPAI Pyramid, you will once again be able to separate the two operations even when both multipliers have more than one digit. By eliminating carry-over numbers, children can calculate rapidly and securely. Slow students like this approach since it eliminates a good deal of the confusion. We recommend this approach for students requiring remedial aid, and for those who can not understand the conventional approach to multi-digit multiplication. Brighter students should not ignore this approach since it will lead them to more rapid calculations.

The only difference in the HYPAI Pyramid and conventional formats is that the order of multiplying the digits is changed. Since the partial products are now written down in a different order, teachers will have to emphasize place value. Students will have to understand why each partial product is placed in its own particular set of columns.

In step A you multiply 9 x 3. Since the answer is 27, it is placed into the tens' and units' columns. Step B is where the format begins to change from the conventional. In step B you multiply 80 x 70. Since the result is 5600, a 5 is placed into the thousands' column and a 6 into the hundreds' column. Step C calls for the multiplication of 9 x 70. Since the result is 630, a 6 is placed in the hundreds' column and a 3 in the tens' column. In step D you multiply 80 x 3. Therefore, you put a 2 in the hundreds' column and a 4 in the tens' column. The final step is to add all the partial products.

| Partial Products | | 7o 3<br>x 8o 9 | Diagram |
|---|---|---|---|
| | | Thousands / Hundreds / Tens / Units | |
| 80 x 70 = 5,600<br>(B) | 9 x 3 = 27<br>(A) | 5 6 \| 2 7<br>(B) (A) | (B) (A) |
| 9 x 70 = 630<br>(C) | | 6 3<br>(C) | |
| 80 x 3 = 240<br>(D) | + | 2 4<br>(D) | (D) (C) |
| | | 6 4 9 7 | |

Care must be taken when you have partial products that have only one digit. You *must* insert zeroes in order to preserve the proper place values. In the problem below, the product of 4 x 2 *must* be written as 08 in step A. Similarly, you write the results of step B as 06, and the results of step D as 02.

The diagrams have been placed on every page of problems. Before starting to calculate, carefully examine the diagram and make sure you know the order you will be using when you are finding your partial products.

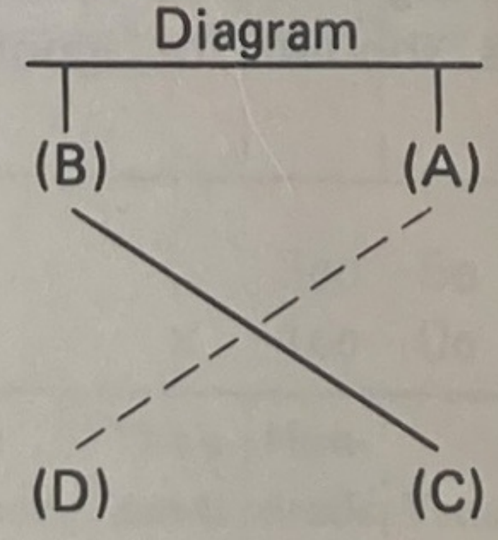
EXAMPLE:

```
    6   2
x   1   4
-------
  0 6 0 8
  (B) (A)

  2 4
  (C)

  0 2
  (D)
-------
  8 6 8
```

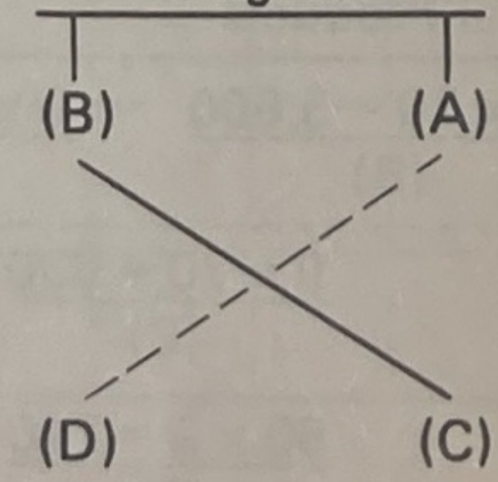
## EXERCISES

Each preliminary step requires a two-digit answer. If zero is implied, write it down to preserve the proper place values for the succeeding steps.

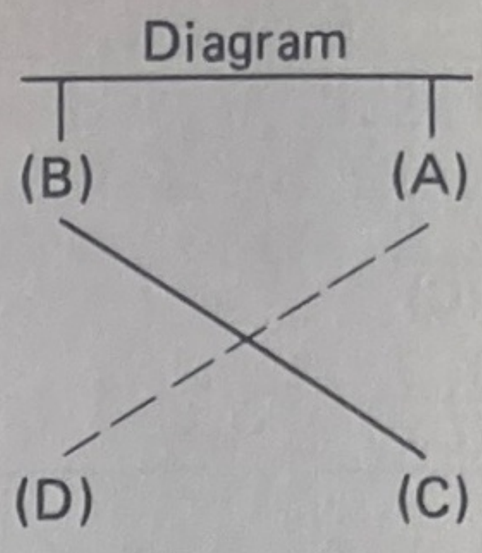
```
    4   3        8   2        3   2        5   3
x   2   1    x   1   9    x   3   4    x   1   2
-------      -------      -------      -------
  0 8 0 3
  (B) (A)
- - - - -    - - - - -    - - - - -    - - - -
    0 4
    (C)
    0 6
+   (D)      +            +            +
-------      -------      -------      -------
```



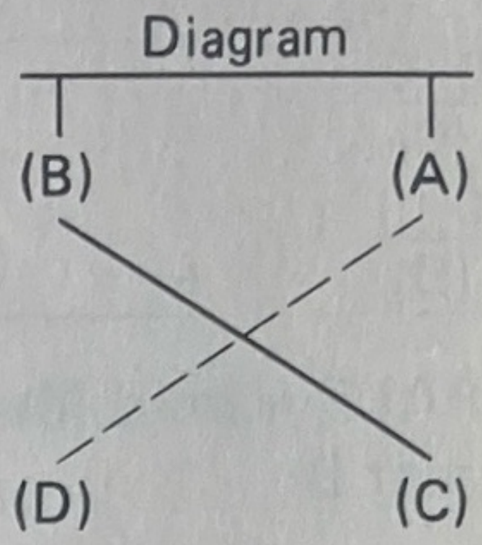

```
    8   7        2   9        5   3        4   5
x   3   6    x   2   4    x   6   3    x   1   2
-------      -------      -------      -------
  2 4 4 2
  (B) (A)
- - - - -    - - - - -    - - - - -    - - - - -
    4 8
    (C)
    2 1
+   (D)      +            +            +
-------      -------      -------      -------
```



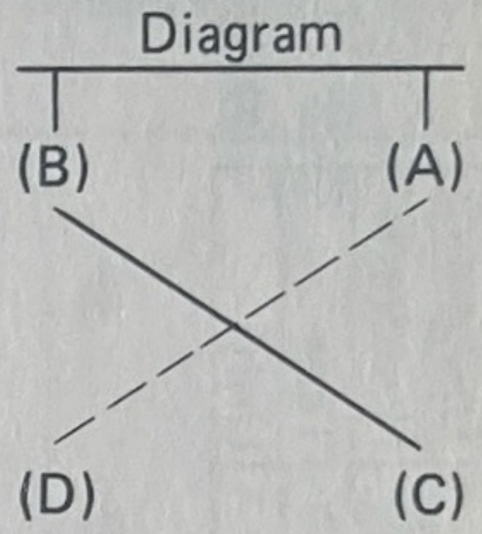
| | | | |
|---|---|---|---|
| 66 | 12 | 90 | 54 |
| x 75 | x 21 | x 64 | x 45 |
| 4230 | | | |
| (B) (A) | | | |
| ———— | ———— | ———— | ———— |
| 30 | | | |
| (C) | | | |
| 42 | | | |
| + (D) | + | + | + |
| ———— | ———— | ———— | ———— |



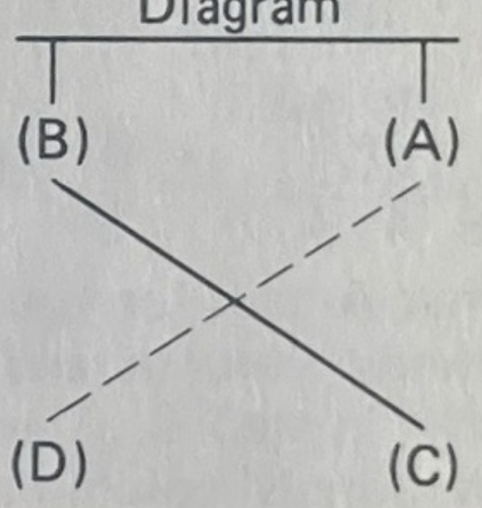

| | | | |
|---|---|---|---|
| 78 | 39 | 11 | 82 |
| x 32 | x 65 | x 75 | x 43 |
| 2116 | | | |
| (B) (A) | | | |
| ———— | ———— | ———— | ———— |
| 14 | | | |
| (C) | | | |
| 24 | | | |
| + (D) | + | + | + |
| ———— | ———— | ———— | ———— |

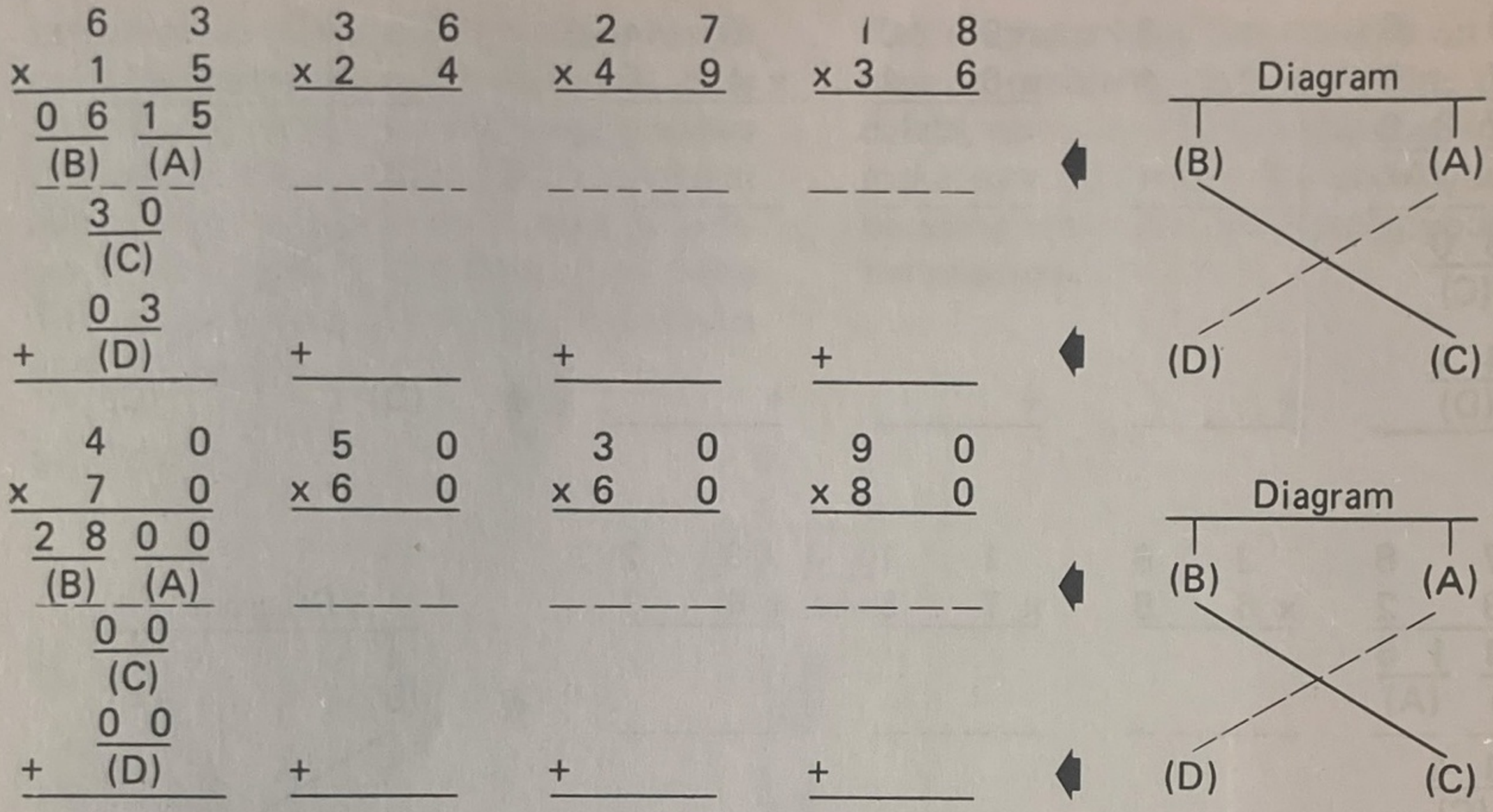



| | | | |
|---|---|---|---|
| 64 | 85 | 92 | 73 |
| x 53 | x 47 | x 56 | x 89 |
| 3012 | | | |
| (B) (A) | | | |
| ———— | ———— | ———— | ———— |
| 18 | | | |
| (C) | | | |
| 20 | | | |
| + (D) | + | + | + |
| ———— | ———— | ———— | ———— |



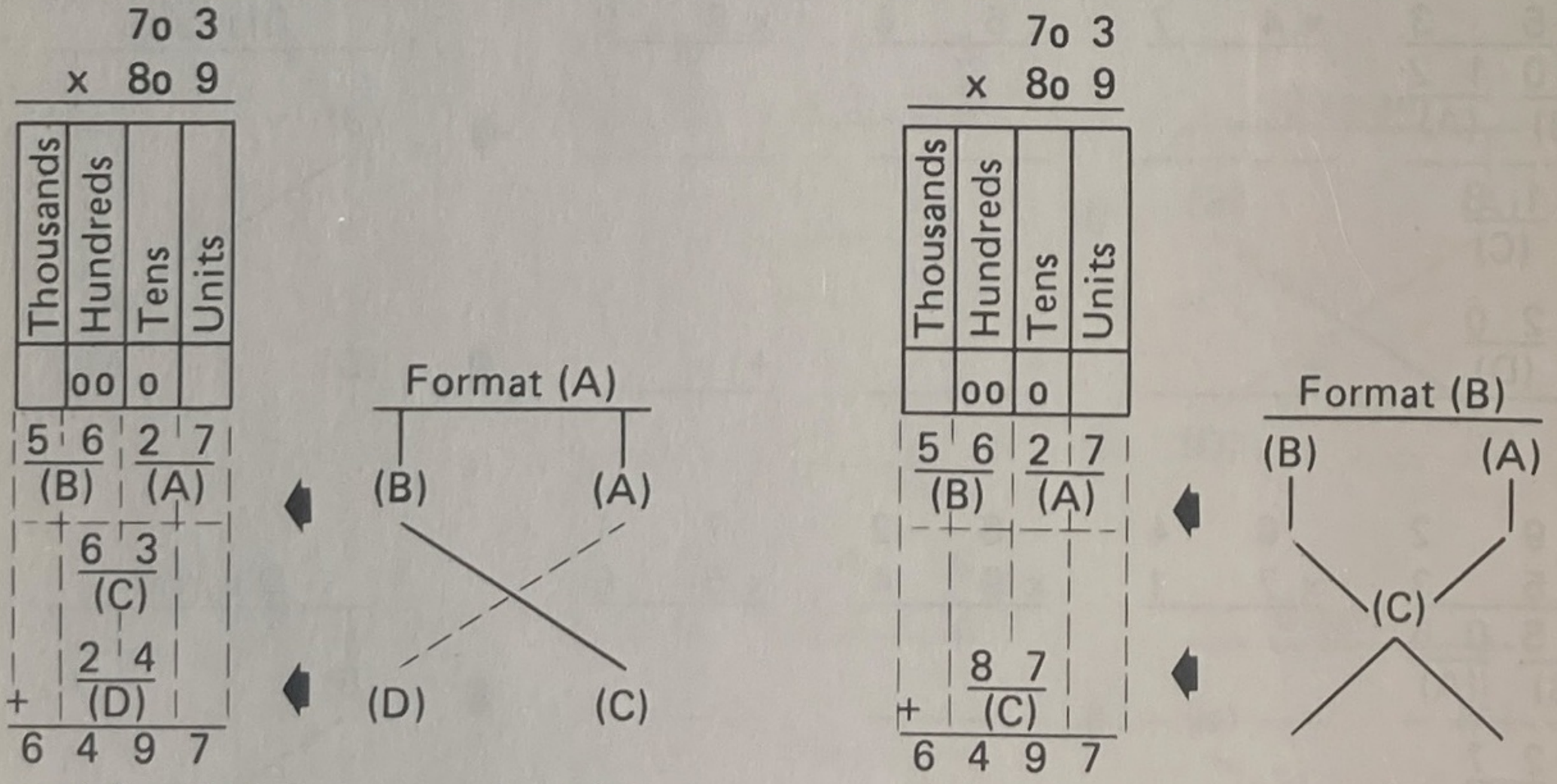

| | | | |
|---|---|---|---|
| 92 | 64 | 82 | 71 |
| x 53 | x 71 | x 94 | x 56 |
| 4506 | | | |
| (B) (A) | | | |
| ———— | ———— | ———— | ———— |
| 27 | | | |
| (C) | | | |
| 10 | | | |
| + (D) | + | + | + |
| ———— | ———— | ———— | ———— |






## HYPAI Pyramid Multiplication Format B

The second format of the HYPAI Pyramid was developed in order to increase the speed of the calculations. Essentially what occurs under the new format is that some of the steps in the old format are converted into single steps. For instance, when you are multiplying two-digit multipliers, steps C and D under the old format become a single step in the new format.




Format A calls for you to write a 6 in the hundreds' column and a 3 in the tens' column in step C. Then you write a 2 in the hundreds' column and a 4 in the tens' column under step D. Format B asks you to add the two partial products, mentally or on the fingers, and only after finding the sum, 87, do you enter an 8 in the hundreds' column and a 7 in the tens' column. You can use the fingers for the addition if you are still most comfortable using them.

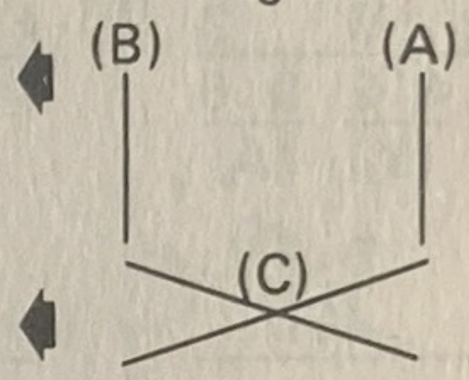
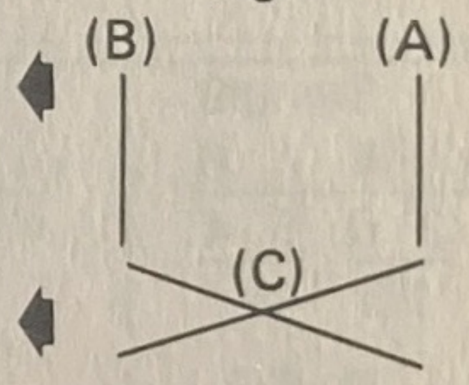
All the other instructions for the Pyramid remain the same. For instance, you must continue to write down the implicit zeroes as seen in the example below. Only by writing down the zeroes can you maintain the proper places for the numbers.

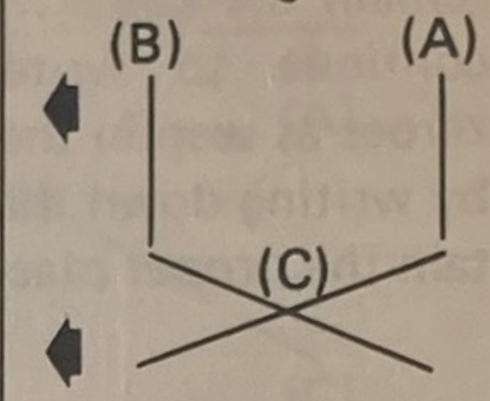
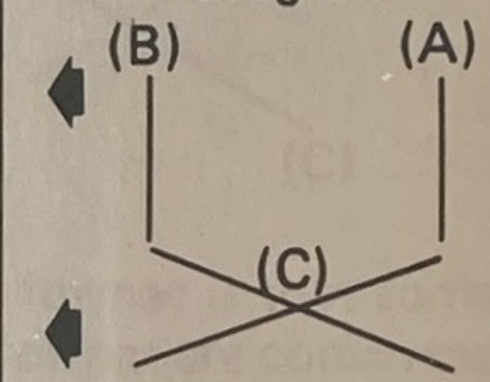
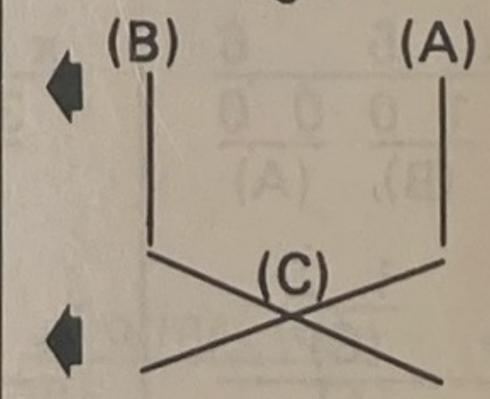
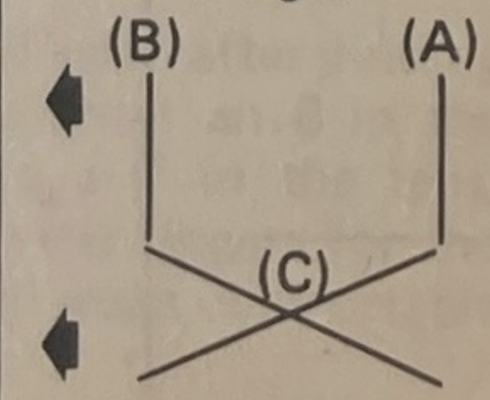
The diagrams have been placed on every page of problems. Before starting to calculate, carefully examine the diagram and make sure you know the order you will be using when you are finding your partial products.

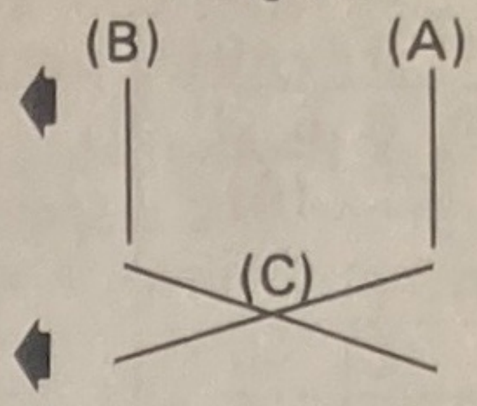
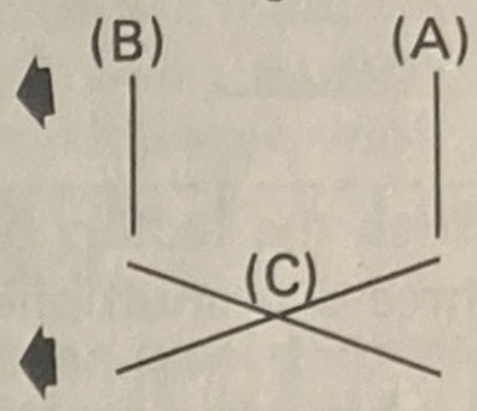
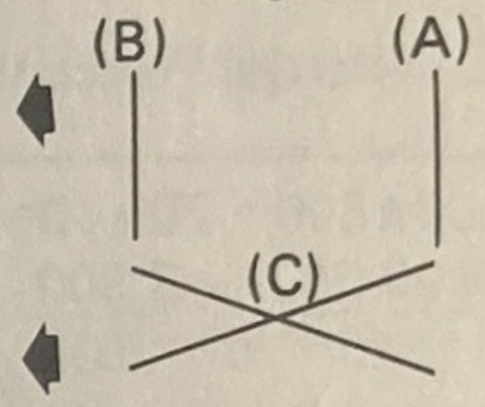
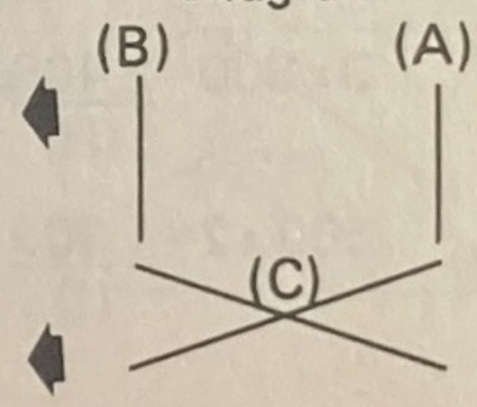
$$\begin{array}{r} 6\quad 2 \\ \times\quad 1\quad 4 \\ \hline 0\ 6\ 0\ 8 \\ \text{(B)}\quad\ \text{(A)} \\ \\ 2\ 6 \\ \text{(C)} \\ \hline 8\ 6\ 8 \end{array}$$

## EXERCISES

Each preliminary step requires a *two-digit answer.* If zero is implied, write it down to preserve the proper place values for the succeeding steps.

| | | | | |
|---|---|---|---|---|
| $\begin{array}{r} 2\quad 0 \\ \times\quad 5\quad 6 \\ \hline 1\ 0\ 0\ 0 \\ \text{(B)}\quad\ \text{(A)} \\ \\ +\quad 1\ 2 \\ \quad\ \text{(C)} \\ \hline \end{array}$ | $\begin{array}{r} 4\quad 6 \\ \times\quad 9\quad 5 \\ \hline 3\ 6\ 3\ 0 \\ \text{(B)}\quad\ \text{(A)} \\ \\ +\quad 7\ 4 \\ \quad\ \text{(C)} \\ \hline \end{array}$ | $\begin{array}{r} 7\quad 0 \\ \times\quad 4\quad 1 \\ \hline 2\ 8\ 0\ 0 \\ \text{(B)}\quad\ \text{(A)} \\ \\ +\quad 0\ 7 \\ \quad\ \text{(C)} \\ \hline \end{array}$ | $\begin{array}{r} 1\quad 4 \\ \times\quad 3\quad 2 \\ \hline 0\ 3\ 0\ 8 \\ \text{(B)}\quad\ \text{(A)} \\ \\ +\quad 1\ 4 \\ \quad\ \text{(C)} \\ \hline \end{array}$ | Diagram   |
| $\begin{array}{r} 7\quad 6 \\ \times\quad 9\quad 3 \\ \hline \\ \\ +\quad\quad \\ \hline \end{array}$ | $\begin{array}{r} 7\quad 1 \\ \times\quad 4\quad 0 \\ \hline \\ \\ +\quad\quad \\ \hline \end{array}$ | $\begin{array}{r} 1\quad 5 \\ \times\quad 7\quad 5 \\ \hline \\ \\ +\quad\quad \\ \hline \end{array}$ | $\begin{array}{r} 2\quad 6 \\ \times\quad 4\quad 9 \\ \hline \\ \\ +\quad\quad \\ \hline \end{array}$ | Diagram   |

| | | | | |
|---|---|---|---|---|
| 1 8<br>x 4 3<br><br>+ | 6 7<br>x 3 6<br><br>+ | 9 3<br>x 2 5<br><br>+ | 8 6<br>x 7 4<br><br>+ | Diagram<br>  <br> |
| 2 9<br>x 4 6<br><br>+ | 5 3<br>x 7 8<br><br>+ | 6 1<br>x 4 2<br><br>+ | 7 0<br>x 3 1<br><br>+ | Diagram<br>  <br> |
| 7 2<br>x 6 3<br>4 2 0 6<br>(B) (A)<br>3 3<br>+ (C) | 4 9<br>x 5 8<br><br>+ | 3 1<br>x 2 7<br><br>+ | 1 6<br>x 4 7<br><br>+ | Diagram<br>  <br> |
| 5 9<br>x 4 6<br>2 0 5 4<br>(B) (A)<br>6 6<br>+ (C) | 1 2<br>x 4 9<br><br>+ | 7 3<br>x 7 3<br><br>+ | 6 0<br>x 4 9<br><br>+ | Diagram<br>  <br> |

| | | | | |
|---|---|---|---|---|
| 2 1<br>x 4 3<br>0 8 0 3<br>(B) (A)<br>1 0<br>+ (C) | 6 9<br>x 5 5 | 7 8<br>x 1 1 | 9 6<br>x 3 0 | Diagram<br>  |
| 6 4<br>x 1 8<br>0 6 3 2<br>(B) (A)<br>5 2<br>+ (C) | 5 4<br>x 1 2 | 9 3<br>x 7 6 | 1 9<br>x 9 1 | Diagram<br>  |
| 6 6<br>x 9 8<br>5 4 4 8<br>(B) (A)<br>1 0 2<br>+ (C) | 6 9<br>x 8 7 | 8 7<br>x 7 8 | 8 9<br>x 5 9 | Diagram<br>  |
| 5 8<br>x 8 9<br>4 0 7 2<br>(B) (A)<br>1 0 9<br>+ (C) | 8 7<br>x 9 5 | 7 7<br>x 9 6 | 9 8<br>x 7 7 | Diagram<br>  |

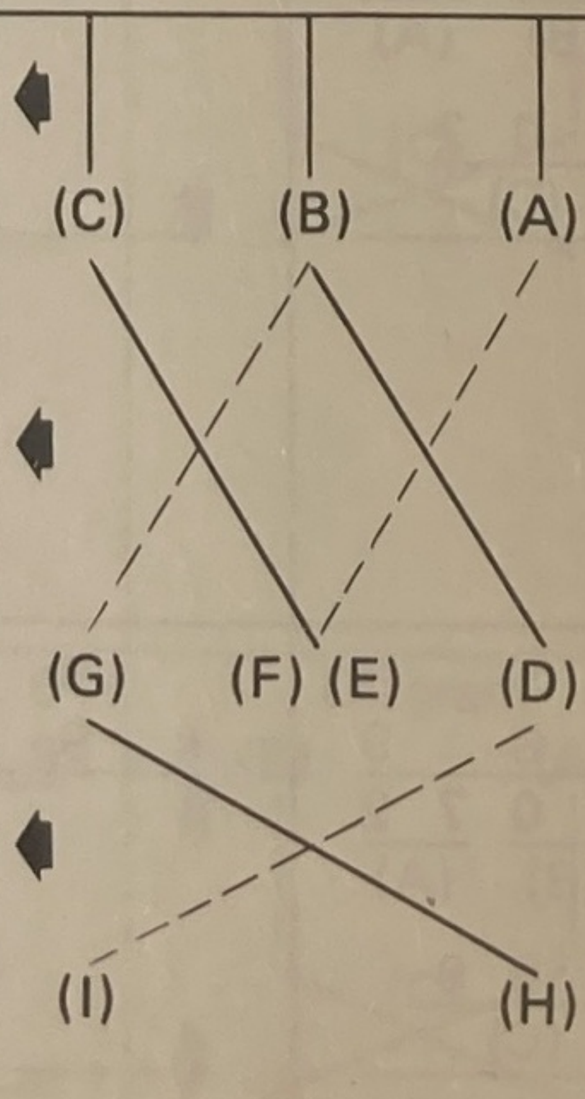
| | | | | |
|---|---|---|---|---|
| 8 7<br>x 6 9<br>4 8 6 3<br>(B) (A)<br>1 1 4<br>+ (C) | 7 8<br>x 8 7<br>+ | 6 8<br>x 7 8<br>+ | 8 9<br>x 5 9<br>+ | Diagram<br>(B) (A)<br>(C) |
| 6 9<br>x 9 6<br>5 4 5 4<br>(B) (A)<br>1 1 7<br>+ (C) | 7 6<br>x 7 9<br>+ | 6 8<br>x 8 7<br>+ | 8 8<br>x 7 6<br>+ | Diagram<br>(B) (A)<br>(C) |

When the factors have three digits, you can also use the HYPAI Pyramid. Format (A) for three-digit multipliers is shown below.

800 40 2
x 500 70 3

| Partial Products | Hundred Thousands | Ten Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|---|---|
| | | 000<br>0 | 000 | 00 | 0 | |

Diagram




500×800= 70×40= 3×2=
400,000 2,800 06
(C) (B) (A)
4 0 2 8 0 6
(C) (B) (A)

70×800= 3×40=
56,000 120
(F) (D)
5 6 1 2
(F) (D)

500×40= 70×2=
20,000 140
(G) (E)
2 0 1 4
(G) (E)

3×800=2,400
(H)
2 4
(H)

500×2=1,000
(I)
1 0
(I)

\+

4 8 2 4 6 6

Step A calls for you to multiply 3 x 2. You therefore write a 6 in the units' column. To preserve place value and keep a consistent format you also write a 0 in the tens' column.

Step B asks you to multiply 70 x 40. Since the answer is 2800, you write a 2 in the thousands' column and an 8 in the hundreds' column.

In step C you have to multiply 500 x 800. The answer is 400,000. Consequently, you write a 4 in the hundred thousands' column and a 0 in the ten thousands' column.

Step D requires you to multiply 3 x 40. You therefore write a 1 in the hundreds column and a 2 in the tens' column.

Step E looks for the product of 70 x 2. Therefore, a 1 must be placed in the hundreds' column and a 4 in the tens' column.

In step F you must multiply 70 x 800. You place a 5 in the ten thousands' column and a 6 in the thousands' column.

Step G shows the product of 500 x 40. A 2 goes in the ten thousands' column and a 0 in the thousands' column.

In step H you must multiply 3 x 800. Consequently, you place a 2 in the thousands' column and a 4 in the hundreds' column.

Step I requires you to multiply 500 x 2. Therefore, you place a 1 in the thousands' column and a 0 in the hundreds' column.

Don't forget to take the final step of adding up all the partial products.

When multiplying a three-digit number by a two-digit number, you use the same format as the one used when multiplying three-digit multipliers. However, you must treat the two-digit number as if it has a zero placed in front of it.

The diagrams have been placed on every page of problems. Before starting to calculate, carefully examine the diagram and make sure you know the order you will be using when you are finding your partial products.

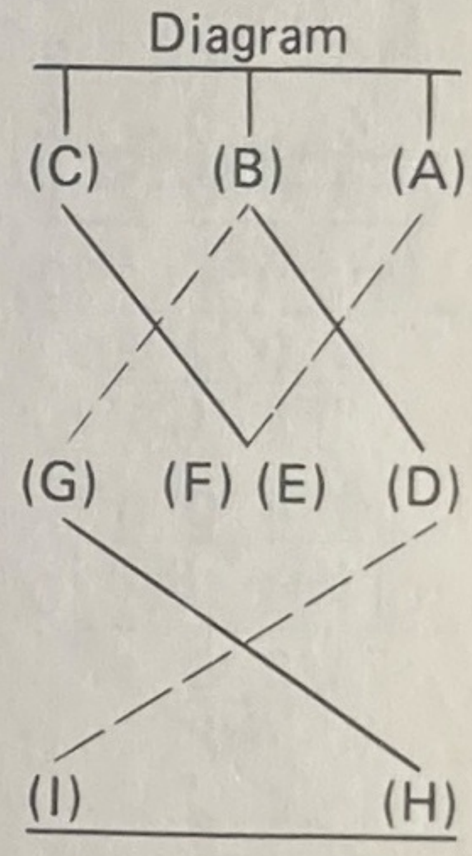
## EXERCISES

Each preliminary step requires a two-digit answer. If zero is implied, write it down to preserve the proper place values for the succeeding steps.

```
    2   7   5
x   4   3   6
-----------
  0 8 2 1 3 0
    (C) (B) (A)
-----------
    0 6 4 2
      (F) (D)
    2 8 1 5
      (G) (E)
-----------
      1 2
      (H)
    2 0
+   (I)
-----------
```

```
    9   1   3
x   4   6   7
-----------
+
-----------
```

```
    8   4   2
x   5   7   3
-----------
+
-----------
```

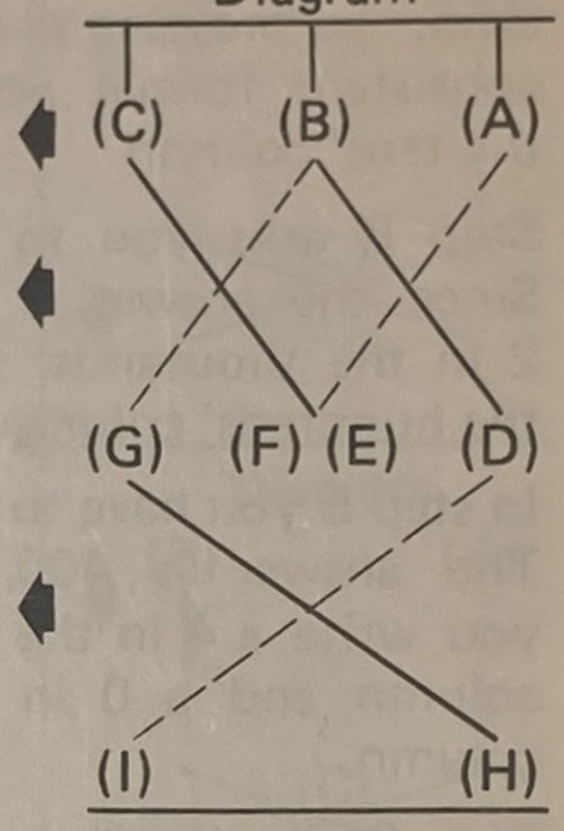


$$\begin{array}{r} 6\ \ 0\ \ 4 \\ \times\ 7\ \ 0\ \ 3 \\ \hline 4\ 2\ \ 0\ 0\ \ 1\ 2 \\ \text{(C)}\quad\text{(B)}\quad\text{(A)} \\ \hline 0\ 0\ \ 0\ 0 \\ \text{(F)}\quad\text{(D)} \\ 0\ 0\ \ 0\ 0 \\ \text{(G)}\quad\text{(E)} \\ 1\ 8 \\ \text{(H)} \\ 2\ 8 \\ +\quad\text{(I)} \\ \hline \end{array}$$

$$\begin{array}{r} 7\ \ 2\ \ 6 \\ \times\ 4\ \ 3\ \ 8 \\ \hline \\ \\ \hline \\ \\ \\ \\ \\ \\ + \\ \hline \end{array}$$

$$\begin{array}{r} 7\ \ 3\ \ 0 \\ \times\ 8\ \ 5\ \ 0 \\ \hline \\ \\ \hline \\ \\ \\ \\ \\ \\ + \\ \hline \end{array}$$

Diagram

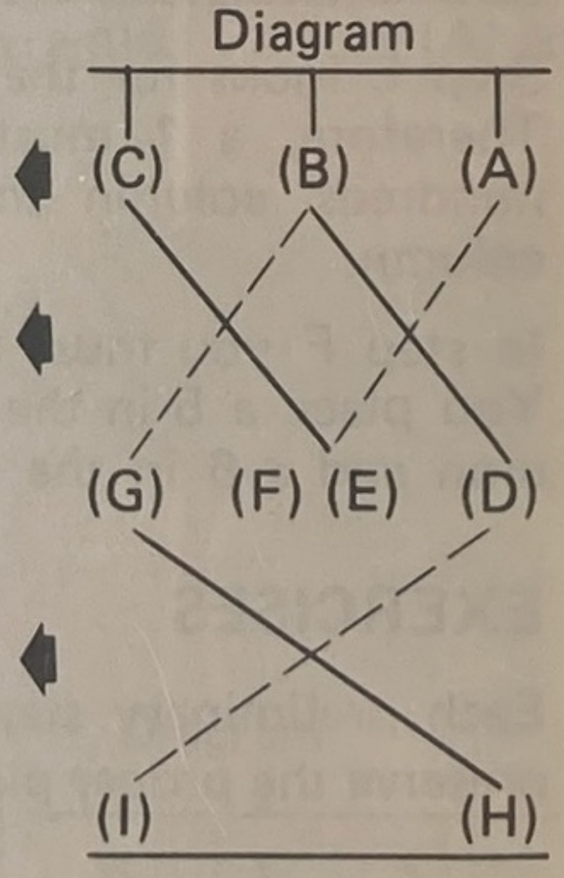



$$\begin{array}{r} 6\ \ 8\ \ 0 \\ \times\ 2\ \ 5\ \ 0 \\ \hline 1\ 2\ \ 4\ 0\ \ 0\ 0 \\ \text{(C)}\quad\text{(B)}\quad\text{(A)} \\ \hline 3\ 0\ \ 0\ 0 \\ \text{(F)}\quad\text{(D)} \\ 1\ 6\ \ 0\ 0 \\ \text{(G)}\quad\text{(E)} \\ 0\ 0 \\ \text{(H)} \\ 0\ 0 \\ +\quad\text{(I)} \\ \hline \end{array}$$

$$\begin{array}{r} 1\ \ 6\ \ 4 \\ \times\ 3\ \ 9\ \ 2 \\ \hline \\ \\ \hline \\ \\ \\ \\ \\ \\ + \\ \hline \end{array}$$

$$\begin{array}{r} 4\ \ 3\ \ 2 \\ \times\ 1\ \ 8\ \ 2 \\ \hline \\ \\ \hline \\ \\ \\ \\ \\ \\ + \\ \hline \end{array}$$

Diagram

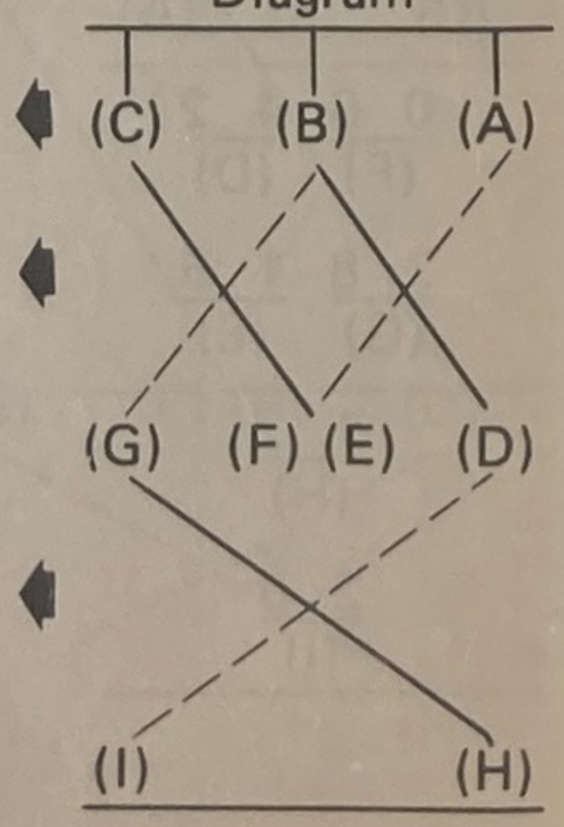



$$\begin{array}{r} 5\ \ 4\ \ 3 \\ \times\ 6\ \ 0\ \ 7 \\ \hline 3\ 0\ \ 0\ 0\ \ 2\ 1 \\ \text{(C)}\quad\text{(B)}\quad\text{(A)} \\ \hline 0\ 0\ \ 2\ 8 \\ \text{(F)}\quad\text{(D)} \\ 2\ 4\ \ 0\ 0 \\ \text{(G)}\quad\text{(E)} \\ 3\ 5 \\ \text{(H)} \\ 1\ 8 \\ +\quad\text{(I)} \\ \hline \end{array}$$

$$\begin{array}{r} 1\ \ 2\ \ 9 \\ \times\ 4\ \ 8\ \ 3 \\ \hline \\ \\ \hline \\ \\ \\ \\ \\ \\ + \\ \hline \end{array}$$

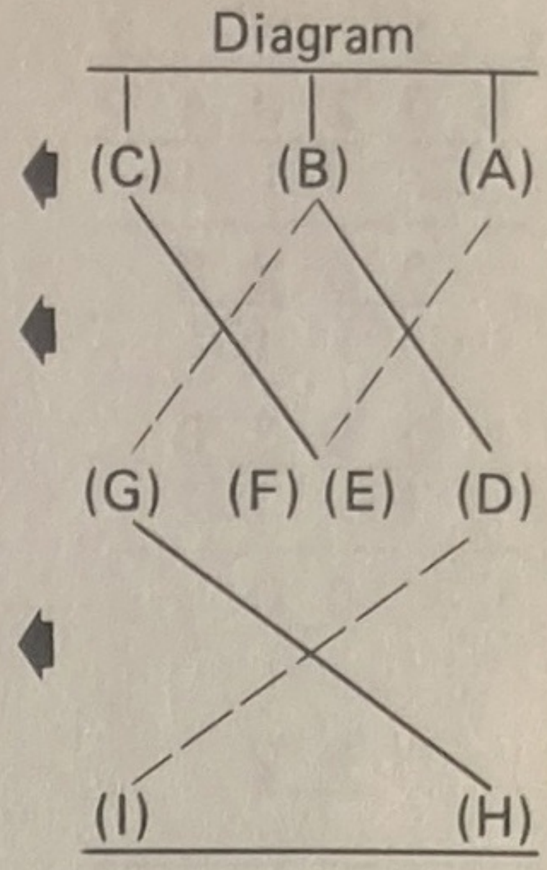
$$\begin{array}{r} 3\ \ 0\ \ 8 \\ \times\ 9\ \ 6\ \ 1 \\ \hline \\ \\ \hline \\ \\ \\ \\ \\ \\ + \\ \hline \end{array}$$

Diagram



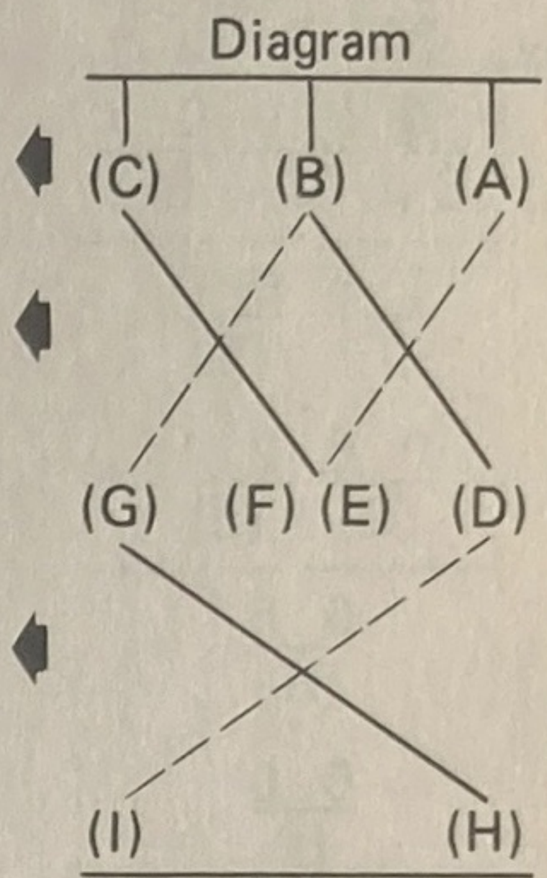
| | | |
|---|---|---|
| 9 2 6<br>x 4 3 7 | 8 0 4<br>x 2 6 7 | 7 1 5<br>x 4 2 3 |
| 3 6 0 6 4 2<br>(C) (B) (A) | | |
| 2 7 1 4<br>(F) (D) | | |
| 0 8 1 8<br>(G) (E) | | |
| 6 3<br>(H) | | |
| 2 4<br>+ (I) | + | + |

Diagram



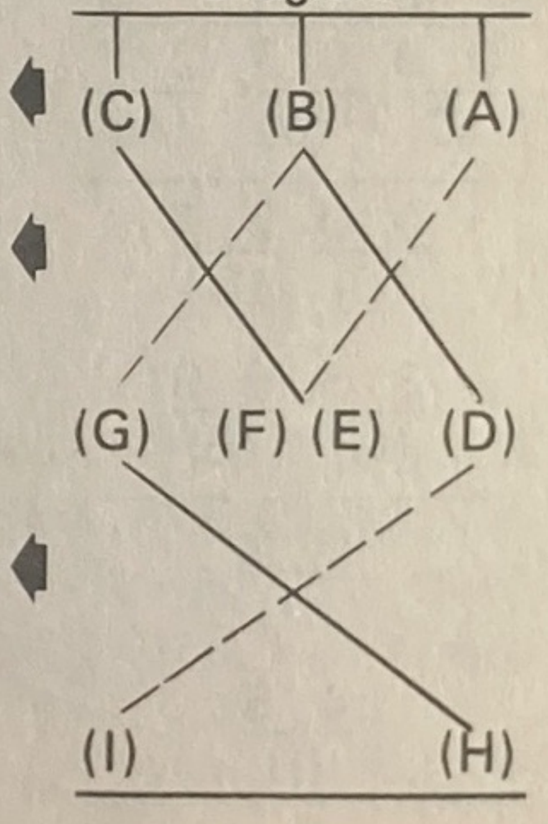

| | | |
|---|---|---|
| 1 5 3<br>x 2 7 3 | 4 1 6<br>x 9 3 5 | 5 7 3<br>x 4 6 2 |
| 0 2 3 5 0 9<br>(C) (B) (A) | | |
| 0 7 1 5<br>(F) (D) | | |
| 1 0 2 1<br>(G) (E) | | |
| 0 3<br>(H) | | |
| 0 6<br>+ (I) | + | + |

Diagram




| | | |
|---|---|---|
| 7 2 4<br>x ◯ 8 6 | 9 3 5<br>x ◯ 4 3 | 5 1 3<br>x ◯ 2 7 |
| 0 0 1 6 2 4<br>(C) (B) (A) | | |
| 5 6 1 2<br>(F) (D) | | |
| 0 0 3 2<br>(G) (E) | | |
| 4 2<br>(H) | | |
| 0 0<br>+ (I) | + | + |

Diagram



( ) 4 6
x 2 5 7

0 0 2 0 4 2
(C) (B) (A)

0 0 2 8
(F) (D)

0 8 3 0
(G) (E)

0 0
(H)

1 2
+ (I)

( ) 8 3
x 6 1 4

( ) 2 3
x 8 1 5

Diagram

(C) (B) (A)

(G) (F) (E) (D)

(I) (H)

5 0 7
x ( ) 3 1

0 0 0 0 0 7
(C) (B) (A)

1 5 0 0
(F) (D)

0 0 2 1
(G) (E)

0 5
(H)

0 0
+ (I)

( ) 7 0
x 8 2 9

6 3 8
x ( ) 5 1

Diagram

(C) (B) (A)

(G) (F) (E) (D)

(I) (H)

( ) 7 9
x 5 1 3

0 0 0 7 2 7
(C) (B) (A)

0 0 2 1
(F) (D)

3 5 0 9
(G) (E)

0 0
(H)

4 5
+ (I)

( ) 4 6
x 8 2 7

( ) 2 5
x 3 0 9

Diagram

(C) (B) (A)

(G) (F) (E) (D)

(I) (H)

```
    1   6   8
x   ◯   9   3
-----------
  0 0 5 4 2 4
    (C) (B) (A)
-----------
    0 9 1 8
      (F) (D)

    0 0 7 2
      (G) (E)
-----------
      0 3
      (H)

      0 0
+     (I)
-----------
```

```
    4   5   1
x   ◯   7   6
-----------
+
-----------
```

```
    9   0   2
x   ◯   8   5
-----------
+
-----------
```

Diagram

(C) (B) (A)

(G) (F) (E) (D)

(I) (H)

```
    ◯   5   4
x   9   0   7
-----------
  0 0 0 0 2 8
    (C) (B) (A)
-----------
    0 0 3 5
      (F) (D)

    4 5 0 0
      (G) (E)
-----------
      0 0
      (H)

      3 6
+     (I)
-----------
```

```
    6   8   3
x   ◯   2   4
-----------
+
-----------
```

```
    ◯   1   9
x   4   8   0
-----------
+
-----------
```

Diagram

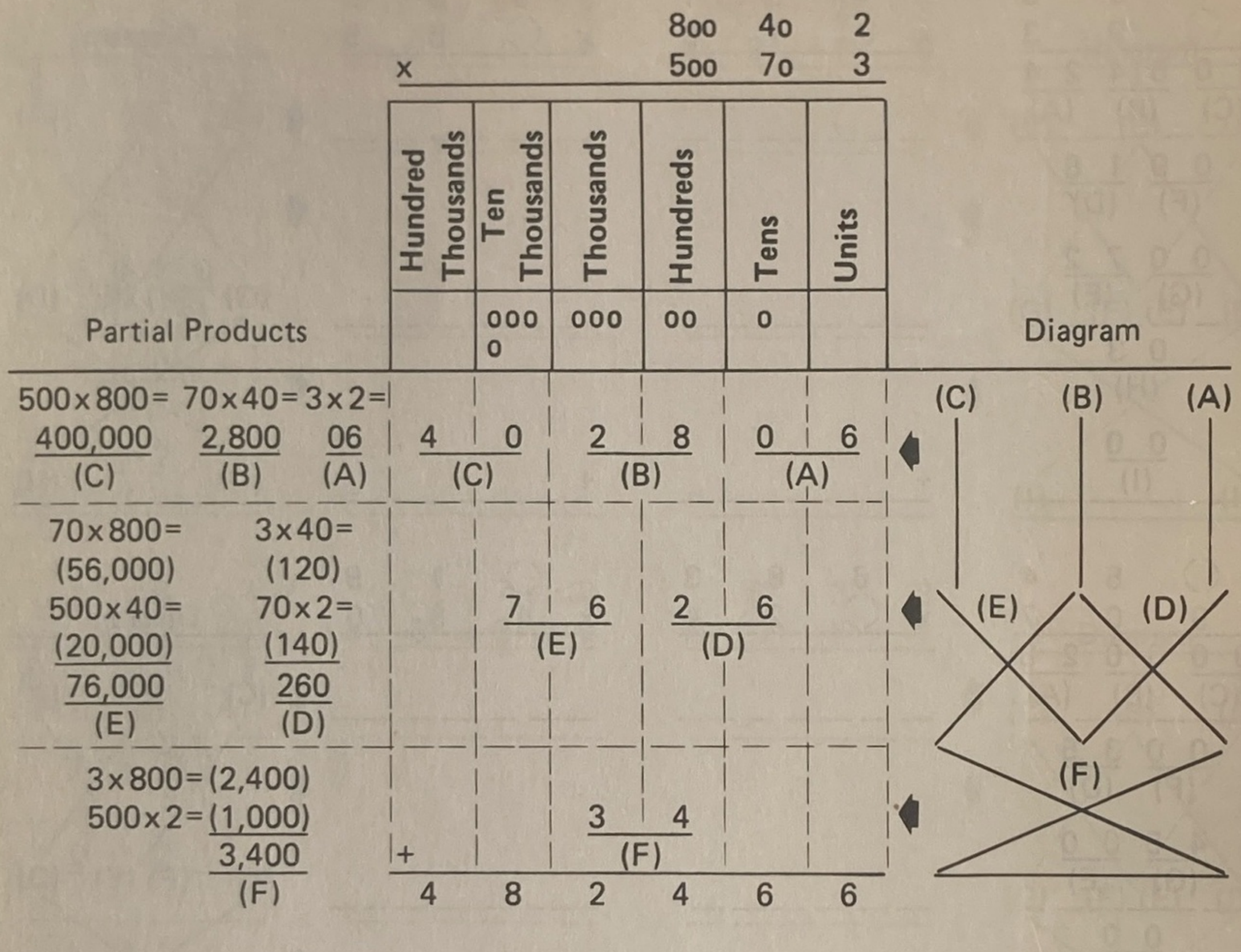
(C) (B) (A)

(G) (F) (E) (D)

(I) (H)

A similar format, B, has been developed for problems that have three-digit multipliers. The new format is displayed below. Steps D & E in the old format have now become step D. We have also combined steps F & G from the old format; they have become step E in the new format. Finally, we have combined steps H & I into step F in the new format.

(FORMAT B)




Let us look more closely at this new format. Steps A, B, & C are exactly the same as they were in the original format. The first change occurs when you reach step D. Here you are asked to find *two* products and to write their *total* down. That is, step D calls for you to find the product of 3 x 40 and the product of 70 x 2. Once you know the products, i.e., 120 and 140, you add the two together and then enter a 2 in the hundreds' column and a 6 in the tens' column.

The entry for step E comes from the product of 70 x 800 and 500 x 40. Since the sum of these two products is 76,000, a 7 is written in the ten thousands' column and a 6 is written in the thousands' column.

Step F calls for the sum of 3 x 800 and 500 x 2. Since these total 3,400, a 3 is written into the thousands' column and a 4 is placed into the hundreds' column. You complete the problem by adding the partial products.

One more case must be shown where the partial products exceed 99. There will be no difficulty as long as you show care in the placement of the right-hand digit.

EXAMPLE:

```
    8   9   7
x   9   8   8
-------------
  7 2 7 2 5 6
  (C) (B) (A)

        1 2 8
  1 4 5 (D)
  (E)

      1 2 7
+     (F)
-------------
  8 8 6 2 3 6
```

The HYPAI Pyramid is seen most clearly whenever the products are limited to two-digit numbers. However, sometimes when combining products, a three-digit number is encountered. If such is the case, we must explain how to handle this subject under two categories.

*Case 1:* Where the student is using his fingers to accumulate the totals, consider the problem below.

```
      7   8   9
x     8   9   7
-------------------
    5 6 7 2 6 3
    1 2 8 3 7
+     1 2 1
-------------------
    7 0 7 7 3 3
```

Using the HYPAI Pyramid we would first write 63 below the seven of the lower multiplier. This would be followed with 72oo immediately to the left of 63; and a 560000 immediately to the left of 72oo. When we come to the second line, we find problems. The first accumulation of products is 137. To begin, the 100 must be stored in the brain and the 37 is written down as a result. Now, when approaching the second set of numbers from the second line, the 1 must be transferred out of the brain and onto the *right* hand. Now we can complete the rest of the second line by adding to the 1, 9x7+8x8. Since this number is also three digits, it must be expressed completely when we write down the answer, 128. The shape of the pyramid is no longer perfect since we are now dealing with accumulations of numbers that exceed two digits.

Again, when we calculate the third line, the answer is three digits. Care must be taken in the placement of the answer, making sure that we move one column to the left. The full answer is found, as in earlier pyramids, by totaling all three lines.

*Case 2:* In this instance, the calculations are performed mentally. When the intermediate calculations are being performed mentally, you can continue in that manner. The only change that must be made occurs when the calculation is not the last one to be done on any given line. Then, only the tens' and units' answers are written down while the hundreds' component is carried over. Once the carry-over is completed, it is then treated as a unit.

Care should be taken that the initial problems given to students using these HYPAI Pyramids do not contain these carry-over complications. Let the students get used to the method before introducing any problems that can distract them. The workbook is very careful along these lines in the selection of problems.

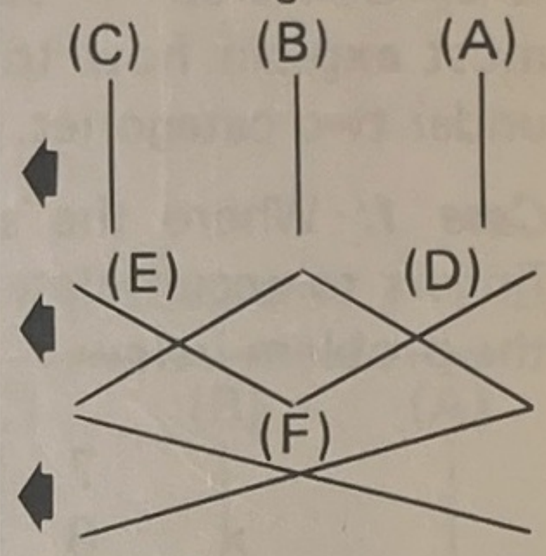
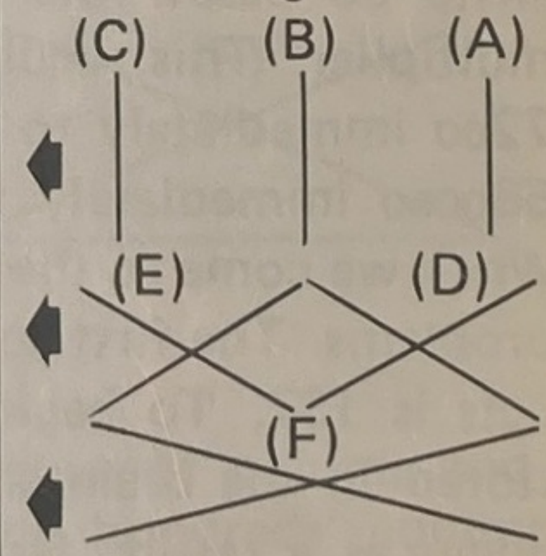
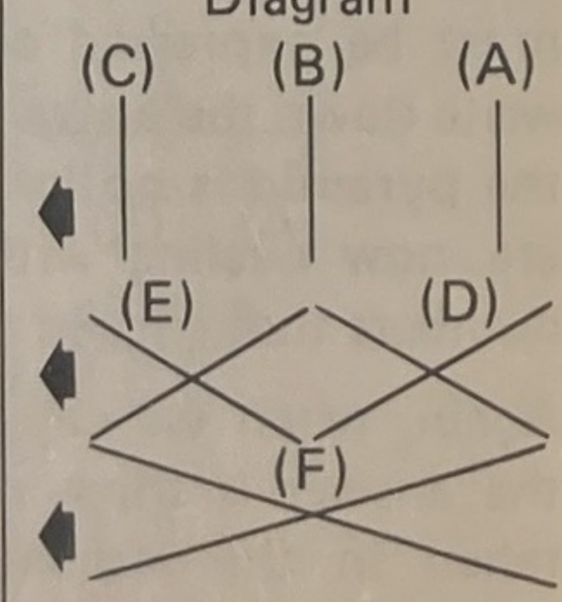
When multiplying a three-digit number by a two-digit number, you use the same format as the one used when multiplying three-digit multipliers. However, you must treat the two-digit number as if it has a zero in front of it.

The diagrams have been placed on every page of problems. Before starting to calculate, carefully examine the diagram and make sure you know the order you will be using when you are finding your partial products.

By the time you have completed all the problems with three-digit multipliers, you should be able to complete the problem with four-digit multipliers from the diagram alone.

## EXERCISES

Each preliminary step requires a two-digit answer. If zero is implied, write it down to preserve the proper place values for the succeeding steps.

| | | |
|---|---|---|
| 6 9 4<br>x 1 5 2<br>0 6 4 5 0 8<br>(C) (B) (A)<br>3 9 3 8<br>(E) (D)<br>1 6<br>+ (F)<br>1 0 5, 4 8 8 | 8 7 5<br>x 2 6 3<br>1 6 4 2 1 5<br>(C) (B) (A)<br>6 2 5 1<br>(E) (D)<br>3 4<br>+ (F) | 4 9 3<br>x 5 0 4<br>2 0 0 0 1 2<br>4 5 3 6<br>3 1<br>+ | Diagram<br>  |
| 5 0 6<br>x 7 0 8<br>+ | 4 6 3<br>x 9 1 0<br>+ | 6 7 8<br>x 1 4 3<br>+ | Diagram<br>  |
| 1 8 0<br>x 9 0 7<br>+ | 8 9 1<br>x 1 5 0<br>+ | 2 6 3<br>x 4 1 8<br>+ | Diagram<br>  |

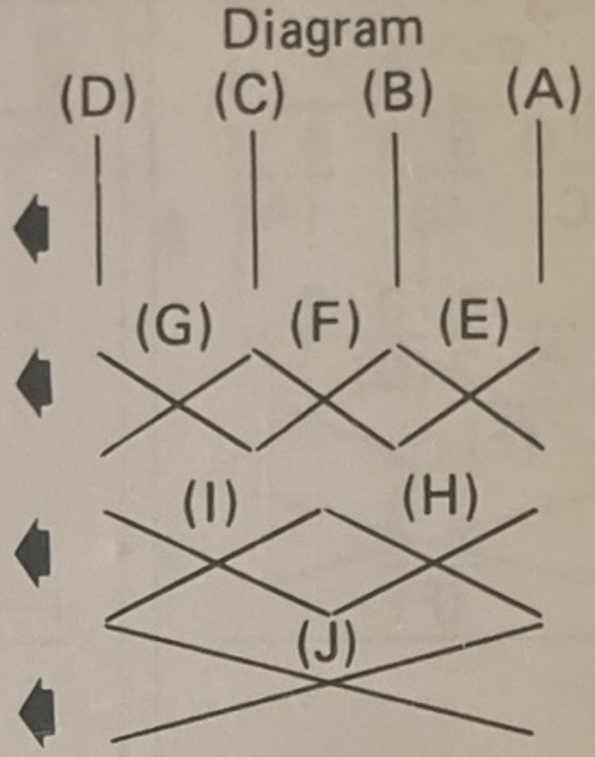
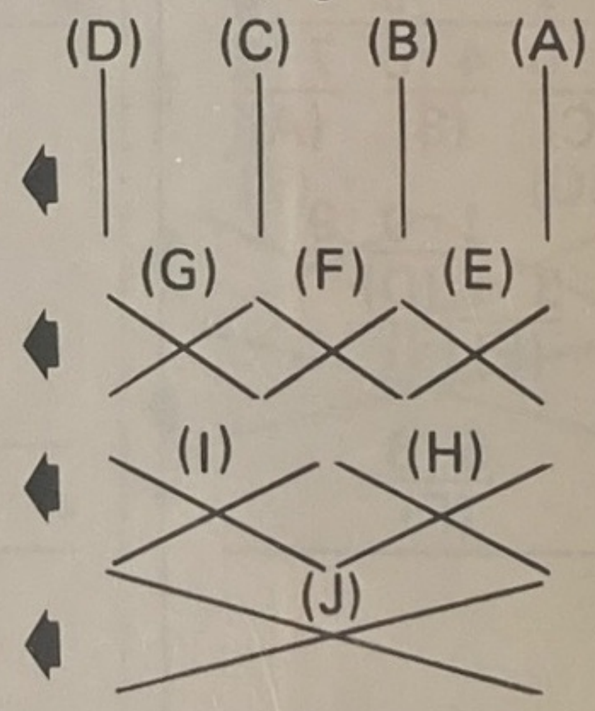
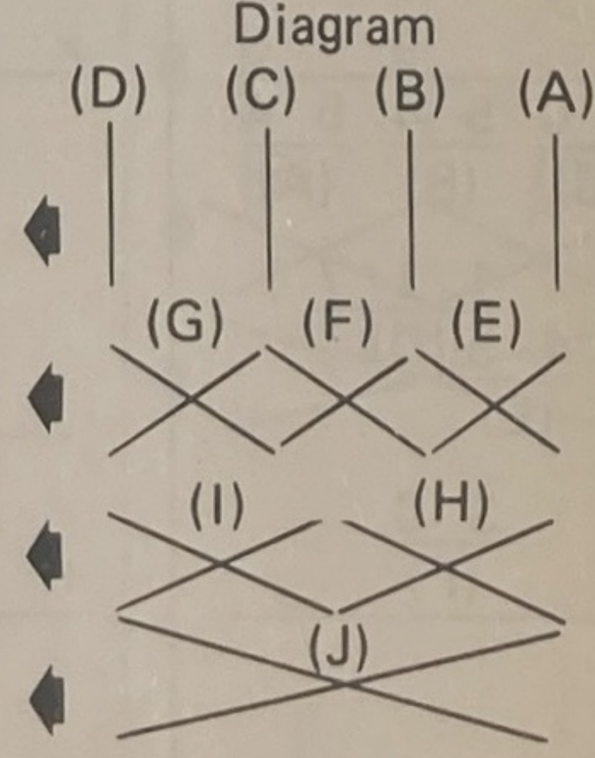
| | | |
|---|---|---|
| 6 4 2<br>x 3 5 1<br>1 8 2 0 0 2<br>(C) (B) (A)<br>4 2 1 4<br>(E) (D)<br>1 2<br>+ (F) | 7 5 3<br>x 1 6 9<br>+ | 2 0 6<br>x 4 7 5<br>+ | Diagram<br>(C) (B) (A)<br>(E) (D)<br>(F) |
| 1 5 6<br>x 7 0 2<br>0 7 0 0 1 2<br>(C) (B) (A)<br>3 5 1 0<br>(E) (D)<br>4 4<br>+ (F) | 3 6 9<br>x 3 0 4<br>+ | 8 3 6<br>x 4 5 7<br>+ | Diagram<br>(C) (B) (A)<br>(E) (D)<br>(F) |
| 7 0 5<br>x 6 4 3<br>4 2 0 0 1 5<br>(C) (B) (A)<br>2 8 2 0<br>(E) (D)<br>5 1<br>+ (F) | 8 1 6<br>x 4 7 0<br>+ | 6 3 1<br>x 1 2 3<br>+ | Diagram<br>(C) (B) (A)<br>(E) (D)<br>(F) |

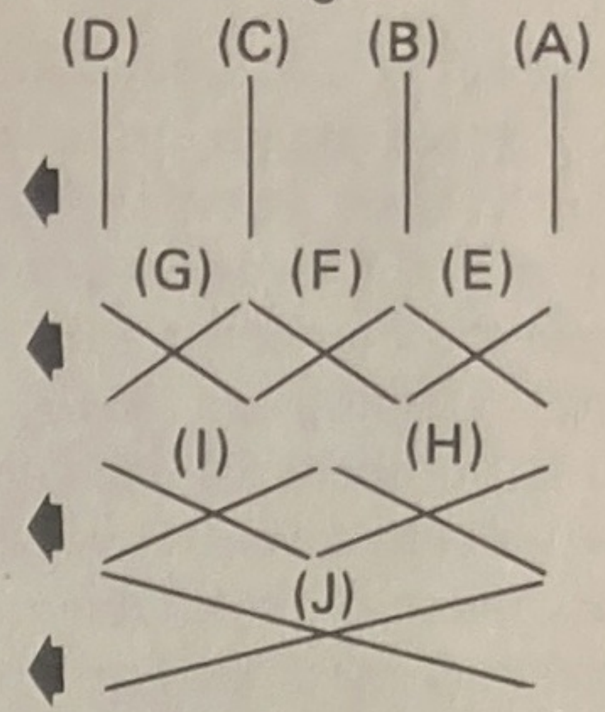
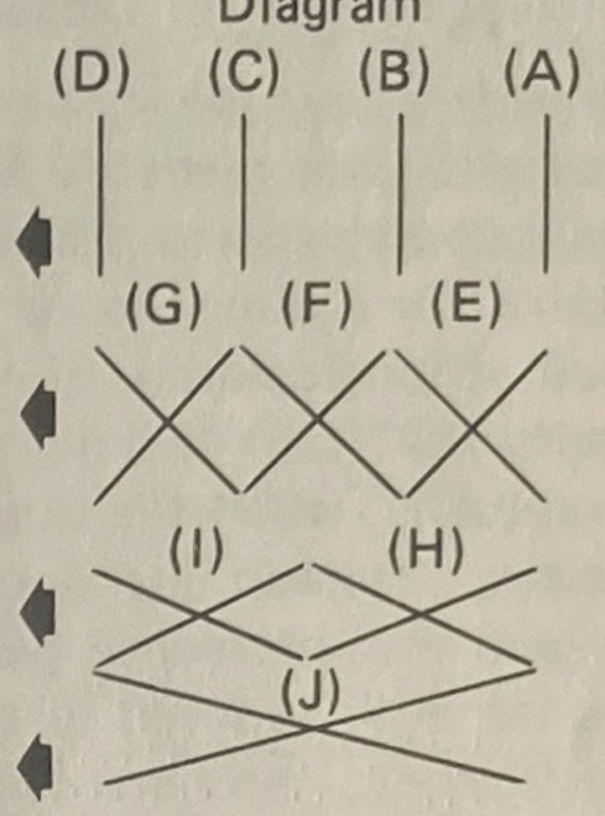
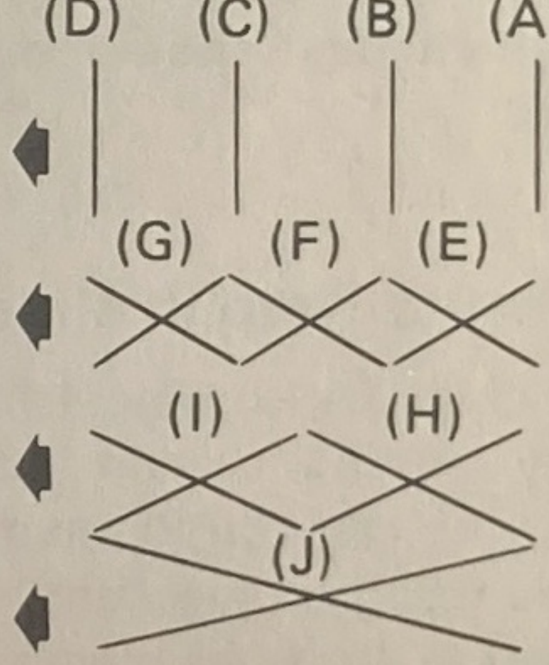
| | | | |
|---|---|---|---|
| 6 0 6<br>x 9 1 8<br>5 4 0 0 4 8<br>(C) (B) (A)<br>0 6 0 6<br>(E) (D)<br>1 0 2<br>+ (F) | 9 2 6<br>x 7 0 9<br>+ | 8 0 9<br>x 9 5 4<br>+ | Diagram<br>(C) (B) (A)<br>(E) (D)<br>(F) |
| 7 5 8<br>x 2 8 9<br>1 4 4 0 7 2<br>(C) (B) (A)<br>1 0 9<br>6 6 (D)<br>(E)<br>7 9<br>+ (F) | 6 8 7<br>x 4 9 5<br>+ | 2 7 7<br>x 3 9 6<br>+ | Diagram<br>(C) (B) (A)<br>(E) (D)<br>(F) |
| 6 9 2<br>x 9 6 4<br>5 4 5 4 0 8<br>(C) (B) (A)<br>1 1 7 4 8<br>(E) (D)<br>4 2<br>+ (F) | 7 6 8<br>x 7 9 3<br>+ | 6 8 1<br>x 8 7 5<br>+ | Diagram<br>(C) (B) (A)<br>(E) (D)<br>(F) |

| | | | |
|---|---|---|---|
| 7 2 0<br>x ◯ 6 4<br>0 0 1 2 0 0<br>(C) (B) (A)<br>4 2 0 8<br>(E) (D)<br>2 8<br>+ (F)<br>4 6 0 8 0 | 9 0 1<br>x ◯ 1 8<br>0 0 0 0 0 8<br>0 9 0 1<br>7 2<br>+ | 4 9 3<br>x ◯ 4 5<br>+ | Diagram<br>(C) (B) (A)<br>(E) (D)<br>(F) |
| ◯ 7 2<br>x 4 3 1<br>0 0 2 1 0 2<br>(C) (B) (A)<br>2 8 1 3<br>(E) (D)<br>0 8<br>+ (F) | ◯ 5 3<br>x 1 2 6<br>0 0 1 0 1 8<br>0 5 3 6<br>0 3<br>+ | ◯ 1 4<br>x 5 9 3<br>+ | Diagram<br>(C) (B) (A)<br>(E) (D)<br>(F) |
| 1 2 4<br>x ◯ 7 9<br>+ | ◯ 8 5<br>x 3 2 6<br>+ | ◯ 9 3<br>x 6 0 4<br>+ | Diagram<br>(C) (B) (A)<br>(E) (D)<br>(F) |

| | | | |
|---|---|---|---|
| 4 ◯ 3 5<br>x 7 2<br>0 0 2 1 1 0<br>(C) (B) (A)<br>2 8 4 1<br>(E) (D)<br>0 8<br>+ (F) | 7 ◯ 3 1<br>x 8 4<br>+ | 5 ◯ 1 6<br>x 7 2<br>+ | Diagram<br>(C) (B) (A)<br>(E) (D)<br>(F) |
| ◯ 6 5<br>x 2 9 4<br>0 0 5 4 2 0<br>(C) (B) (A)<br>1 2 6 9<br>(E) (D)<br>1 0<br>+ (F) | ◯ 3 2<br>x 9 1 7<br>+ | ◯ 7 8<br>x 1 6 3<br>+ | Diagram<br>(C) (B) (A)<br>(E) (D)<br>(F) |
| 1 6 3<br>x ◯ 8 4<br>0 0 4 8 1 2<br>(C) (B) (A)<br>0 8 4 8<br>(E) (D)<br>0 4<br>+ (F) | 7 0 8<br>x ◯ 4 1<br>+ | ◯ 3 4<br>x 5 2 6<br>+ | Diagram<br>(C) (B) (A)<br>(E) (D)<br>(F) |

| | | | |
|---|---|---|---|
| 3 ◯ 6<br>x 9 8<br>0 0 5 4 4 8<br>(C) (B) (A)<br>1 0 2<br>2 7 (D)<br>(E)<br>2 4<br>+ (F) | 5 8 7<br>x ◯ 7 9<br>+ | 5 1 6<br>x ◯ 9 4<br>+ | Diagram<br>(C) (B) (A)<br>(E) (D)<br>(F) |
| ◯ 5 8<br>x 7 8 9<br>0 0 4 0 7 2<br>(C) (B) (A)<br>1 0 9<br>3 5 (D)<br>(E)<br>5 6<br>+ (F) | ◯ 7 7<br>x 1 9 6<br>+ | ◯ 5 7<br>x 3 9 8<br>+ | Diagram<br>(C) (B) (A)<br>(E) (D)<br>(F) |
| 5 6 9<br>x ◯ 9 6<br>0 0 5 4 5 4<br>(C) (B) (A)<br>1 1 7<br>4 5 (D)<br>(E)<br>3 0<br>+ (F) | ◯ 7 6<br>x 6 7 9<br>+ | 6 8 8<br>x ◯ 7 6<br>+ | Diagram<br>(C) (B) (A)<br>(E) (D)<br>(F) |

| | | |
|---|---|---|
| 2, 4 0 6<br>x 5, 3 7 1<br>1 0 1 2 0 0 0 6<br>(D) (C) (B) (A)<br>2 6 2 8 4 2<br>(G) (F) (E)<br>1 4 2 2<br>(I) (H)<br>3 2<br>+ (J)<br>1 2, 9 2 2, 6 2 6 | 6, 9 2 4<br>x 1, 3 5 7<br>0 6 2 7 1 0 2 8<br>(D) (C) (B) (A)<br>2 7 5 1 3 4<br>(G) (F) (E)<br>3 2 7 5<br>(I) (H)<br>4 6<br>+ (J) | Diagram<br>  <br> |
| 4, 1 6 6<br>x 2, 3 0 5<br>+ | 7, 2 1 4<br>x 4, 6 3 8<br>+ | Diagram<br>  <br> |
| 1, 5 2 6<br>x 7, 4 3 8<br>0 7 2 0 0 6 4 8<br>(D) (C) (B) (A)<br>3 9 2 3 3 4<br>(G) (F) (E)<br>1 7 6 4<br>(I) (H)<br>5 0<br>+ (J)<br>1 1, 3 5 0, 3 8 8 | 9, 0 6 7<br>x 8, 4 3 1<br>+ | Diagram<br>  <br> |

| | | |
|---|---|---|
| 4, 9 0 7<br>x 3, 5 2 6<br>1 2 4 5 0 0 4 2<br>(D) (C) (B) (A)<br>4 7 1 8 1 4<br>(G) (F) (E)<br>0 8 8 9<br>(I) (H)<br>4 5<br>+ (J) | 3, 7 6 4<br>x 8, 0 2 9<br>+ | Diagram<br>  <br> |
| 3, 6 6 2<br>x 5, 9 8 4<br>1 5 5 4 4 8 0 8<br>(D) (C) (B) (A)<br>1 0 2 4 0<br>5 7 (F) (E)<br>(G)<br>5 4 4 2<br>(I) (H)<br>2 2<br>+ (J) | 2, 1 6 9<br>x 5, 3 9 7<br>+ | Diagram<br>  <br> |
| 8, 7 2 6<br>x 6, 9 1 5<br>4 8 6 3 0 2 3 0<br>(D) (C) (B) (A)<br>1 1 4 2 5 1 6<br>(G) (F) (E)<br>2 0 8 9<br>(I) (H)<br>7 6<br>+ (J) | 7, 8 0 4<br>x 8, 7 5 1<br>+ | Diagram<br>  <br> |

## ADDITION STAGE 4

Employing the foundation built in the first three stages, in adding columns in fourth stage addition, we add mentally the digits in two successive rows and then press the combined sum. Thus in a problem involving six rows, we press only three numbers for each column. (If there were eight rows, it would be four numbers, and so on.) Obviously, this greatly increases the speed at which calculations are made. Since onety-eight (eighteen) is the largest number obtainable when we add two digits, we now use 20—when 20 is available—as the base around which we build our system of exchanges.

Children must be drilled frequently so that they can mentally combine all two-digit combinations practically automatically. They also should be required to look at long columns of numbers and add orally all the two-digit combinations. Eventually, when the pupils are very proficient, they can abandon the use of their fingers and internalize the entire process. As the drills proceed to this final peak of proficiency, teachers can speed matters along by saying one-eight instead of onety-eight or eighteen, or one three instead of onety-three or thirteen, etc. As individual columns get longer and totals accumulate to over one hundred, the hundreds digit should be stored in one's head.

### Points to Keep in Mind

If a child, in learning 4th stage, has difficulty adding certain number combinations, he can add the numbers separately, as in the 3rd stage until he is more proficient in 4th stage manipulations.

Illustration

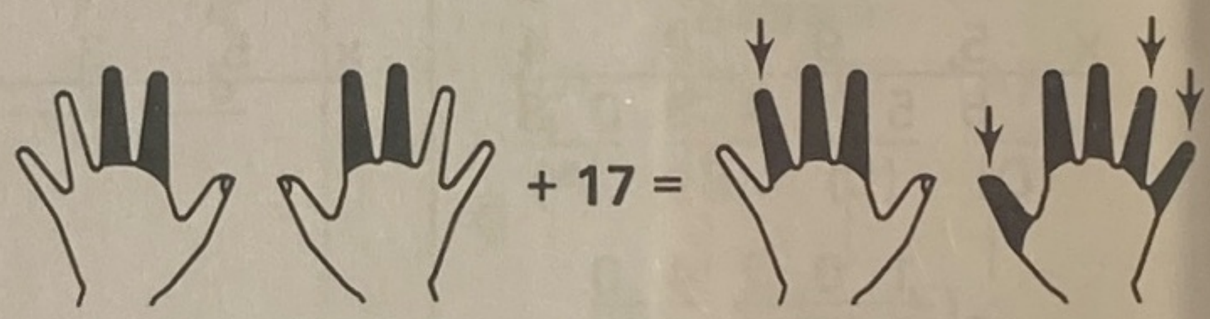
Example A: (Adding the number 17)

The following cases illustrate three ways of adding the number 17.

*Case 1: (+17):* The required fingers are available for pressing. When 17 is added to the numbers 10, 20, 11, 12, 21, 22, 31 or 32, etc. (already pressed) the fingers required to add on 17 are available.

*22 is already pressed*

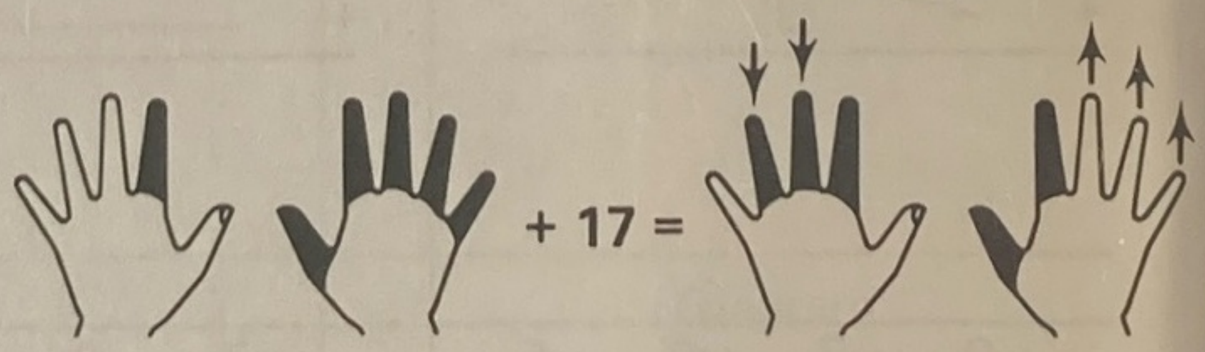
(Press 10 and simultaneously press 7 available fingers on the right hand.)




*Case 2: (20–3)*

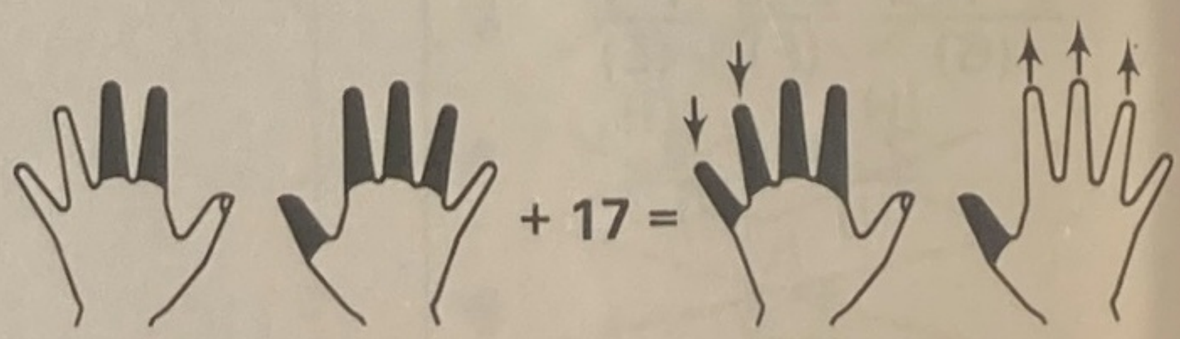
(I) *19 is already pressed*

(Press 20, Clear 3 simultaneously)



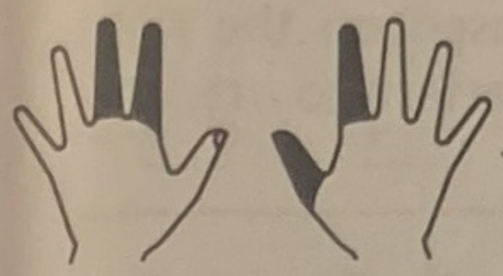

(II) *28 is already pressed*

(Press 20, Clear 3 simultaneously)

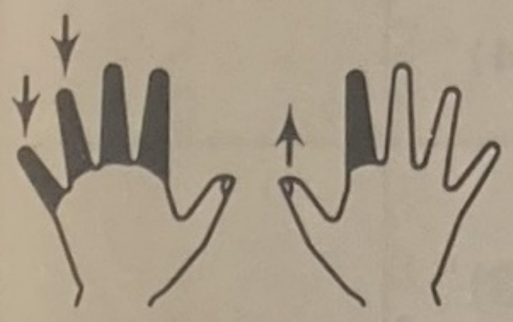


*Case 3: (Exchange 15+2)*

(I) *26 is already pressed*

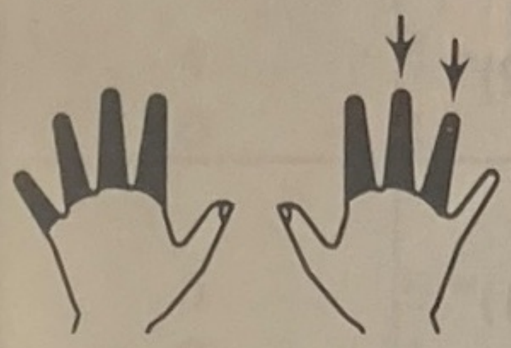



A.



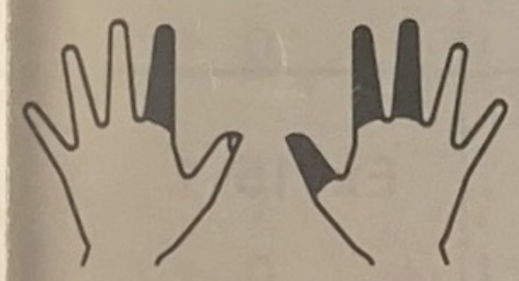
Exchange 15 (20–5) through the manipulation of pressing 20, clearing 5.

B.

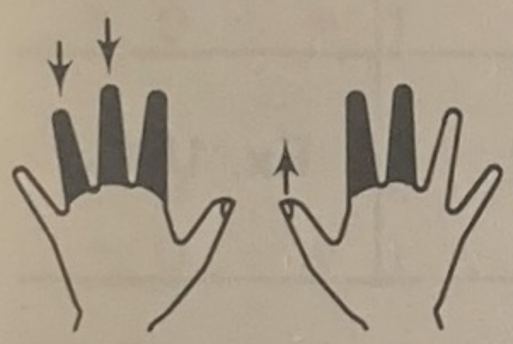


and simultaneously pressing 2 more.

(II) *17 is already pressed*




A.



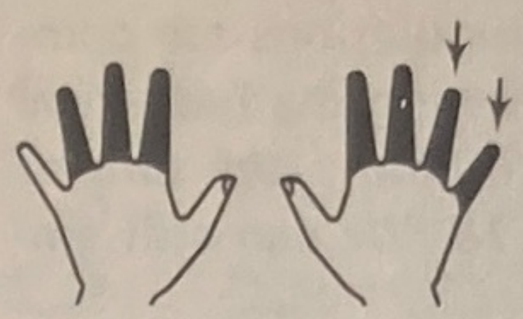
Exchange 15 (20–5) through the manipulation of pressing 20, clearing 5.

B.



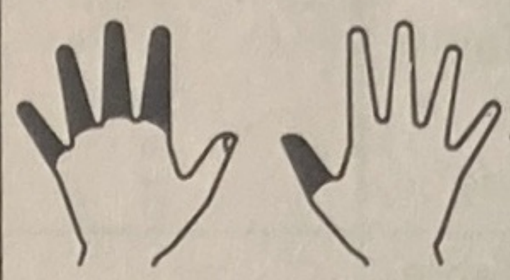
and simultaneously pressing 2 more.

B.

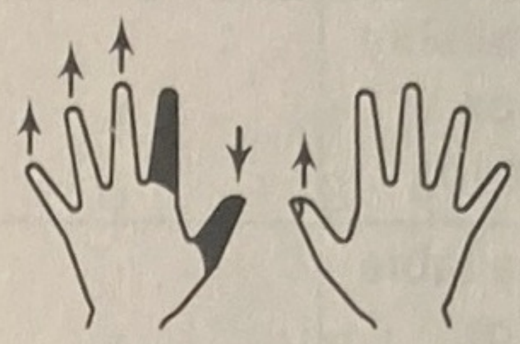


and simultaneously pressing 2 more.

(III) *45 is already pressed*

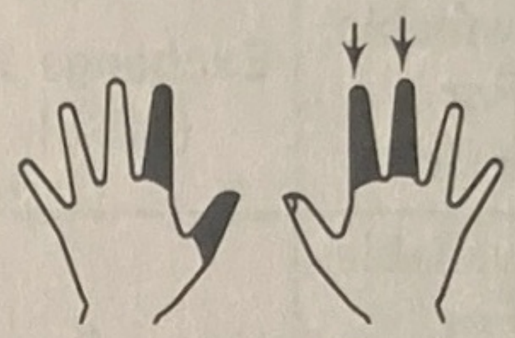



A.



Exchange 15 (20–5) through the manipulation of Left Hand—Press 50 & Clear 30 Right Hand—Clear 5

B.



and simultaneously pressing 2 more.

## ADDITION STAGE 4 PATTERN CHART

Addition Stage 4 manipulations are combinations of the following mathematical patterns employed in *adding the sum of two numbers—11 to 18.* We can still employ the 3rd stage addition manipulation for the sum of two numbers less than 10. The fingers already pressed on the right hand determine which pattern to use.

| Addend \ Pattern | Pattern 1 | Pattern 2 | Pattern 3 | Pattern 4 | Pattern 5 |
|---|---|---|---|---|---|
| 11 | 11 Available For Pressing | | 15-4 | 20-9 (Ex. 15-4)* | |
| 12 | 12 Available For Pressing | | 15-3 | 20-8 (Ex. 15-3)* | |
| 13 | 13 Available For Pressing | | 15-2 | 20-7 (Ex. 15-2)* | |
| 14 | 14 Available For Pressing | | 15-1 | 20-6 (Ex. 15-1)* | |
| 15 | 15 Available For Pressing | Exchange 15 (20-5) | | | |
| 16 | 16 Available For Pressing | | | 20-4 | Ex. 15+1 |
| 17 | 17 Available For Pressing | | | 20-3 | Ex. 15+2 |
| 18 | 18 Available For Pressing | | | 20-2 | Ex. 15+3 |

*The instructions in the parentheses can be used as a temporary manipulation until the student gets used to the more sophisticated maneuver as instructed above the parentheses.

# ADDITION STAGE 4

Mental Calculation Drill

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| + 0 | + 1 | + 2 | + 3 | + 4 | + 5 | + 6 | + 7 | + 8 | + 9 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| + 0 | + 1 | + 2 | + 3 | + 4 | + 5 | + 6 | + 7 | + 8 | + 9 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| + 0 | + 1 | + 2 | + 3 | + 4 | + 5 | + 6 | + 7 | + 8 | + 9 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| + 0 | + 1 | + 2 | + 3 | + 4 | + 5 | + 6 | + 7 | + 8 | + 9 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| + 0 | + 1 | + 2 | + 3 | + 4 | + 5 | + 6 | + 7 | + 8 | + 9 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| + 0 | + 1 | + 2 | + 3 | + 4 | + 5 | + 6 | + 7 | + 8 | + 9 |
| 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| + 0 | + 1 | + 2 | + 3 | + 4 | + 5 | + 6 | + 7 | + 8 | + 9 |
| 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| + 0 | + 1 | + 2 | + 3 | + 4 | + 5 | + 6 | + 7 | + 8 | + 9 |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| + 0 | + 1 | + 2 | + 3 | + 4 | + 5 | + 6 | + 7 | + 8 | + 9 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |
| + 0 | + 1 | + 2 | + 3 | + 4 | + 5 | + 6 | + 7 | + 8 | + 9 |

# EXERCISES

## Pattern 1 & Pattern 1

| | |
|---|---|
| 6 Press 10 then<br>5 Press 1 | 2 Press 10 then<br>9 Press 1 |
| 8 Press 10 then<br>+ 8 Press 6 | 8 Press 10 then<br>+ 7 Press 5 |
| 3 Press 10 then<br>9 Press 2 | 9 Press 10 then<br>4 Press 3 |
| 7 Press 10 then<br>+ 5 Press 2 | 9 Press 10 then<br>+ 7 Press 6 |
| 3 Press 10 then<br>8 Press 1 | 4 Press 10 then<br>9 Press 3 |
| 9 Press 10 then<br>+ 4 Press 3 | 8 Press 10 then<br>+ 3 Press 1 |
| 4 Press 10 then<br>8 Press 2 | 7 Press 10 then<br>9 Press 6 |
| 9 Press 10 then<br>+ 8 Press 7 | 4 Press 10 then<br>+ 8 Press 2 |

## Pattern 1 & Pattern 2

| | |
|---|---|
| 6 Press 10 then<br>9 Press 5 | 7 Press 10 then<br>9 Press 6 |
| 6 Press 20 then<br>+ 9 Clear 5 | 6 Press 20 then<br>+ 9 Clear 5 |
| 8 Press 10 then<br>9 Press 7 | 9 Press 10 then<br>9 Press 8 |
| 8 Press 20 then<br>+ 7 Clear 5 | 9 Press 20 then<br>+ 6 Clear 5 |
| 8 Press 10 then<br>8 Press 6 | 9 Press 10 then<br>7 Press 6 |
| 7 Press 20 then<br>+ 8 Clear 5 | 9 Press 20 then<br>+ 6 Clear 5 |
| 9 Press 10 then<br>8 Press 7 | 9 Press 10 then<br>6 Press 5 |
| 7 Press 20 then<br>+ 8 Clear 5 | 9 Press 20 then<br>+ 6 Clear 5 |

## Pattern 1 & Pattern 3

| | |
|---|---|
| 6 Press 10 then<br>5 Press 1 | 9 Press 10 then<br>+ 5 Press 5, Clear 1 |
| 6 Press 10 then<br>6 Press 2 | 6 Press 10 then<br>+ 7 Press 5, Clear 2 |
| 8 Press 10 then<br>6 Press 4 | 7 Press 10 then<br>+ 7 Press 5, Clear 1 |
| 6 Press 10 then<br>8 Press 4 | 7 Press 10 then<br>+ 6 Press 5, Clear 2 |
| 7 Press 10 then<br>5 Press 2 | 8 Press 10 then<br>+ 6 Press 5, Clear 1 |
| 5 Press 10 then<br>8 Press 3 | 8 Press 10 then<br>+ 5 Press 5, Clear 2 |
| 5 Press 10 then<br>7 Press 2 | 5 Press 10 then<br>+ 8 Press 5, Clear 2 |
| 8 Press 10 then<br>6 Press 4 | 8 Press 10 then<br>+ 6 Press 5, Clear 1 |

Pattern 1 & Pattern 4

| | | | |
|---|---|---|---|
| 5 | Press 10 then | 6 | Press 10 then |
| 7 | Press 2 | 6 | Press 2 |
| 9 | Press 20 then | 9 | Press 20 then |
| + 9 | Clear 2 | + 9 | Clear 2 |

| | | | |
|---|---|---|---|
| 7 | Press 10 then | 5 | Press 10 then |
| 7 | Press 4 | 8 | Press 3 |
| 8 | Press 20 then | 9 | Press 20 then |
| + 9 | Clear 3 | + 8 | Clear 3 |

| | | | |
|---|---|---|---|
| 9 | Press 10 then | 7 | Press 10 then |
| 5 | Press 4 | 9 | Press 6 |
| 8 | Press 20 then | 9 | Press 20 then |
| + 8 | Clear 4 | + 5 | Clear 6 |

| | | | |
|---|---|---|---|
| 9 | Press 10 then | 9 | Press 10 then |
| 8 | Press 7 | 9 | Press 8 |
| 6 | Press 20 then | 6 | Press 20 then |
| + 7 | Clear 7 | + 6 | Clear 8 |

Pattern 1 & Pattern 5

*Exchange 15 means "Press 20, Clear 5 simultaneously."

| | | | |
|---|---|---|---|
| 9 | Press 10 then | 7 | Press 10 then |
| 6 | Press 5 | 9 | Press 6 |
| 7 | Exchange 15 | 9 | Exchange 15 |
| + 9 | then Press 1 | + 7 | then Press 1 |

| | | | |
|---|---|---|---|
| 8 | Press 10 then | 9 | Press 10 then |
| 9 | Press 7 | 9 | Press 8 |
| 8 | Exchange 15 | 7 | Exchange 15 |
| + 8 | then Press 1 | + 9 | then Press 1 |

| | | | |
|---|---|---|---|
| 6 | Press 10 then | 9 | Press 10 then |
| 9 | Press 5 | 8 | Press 7 |
| 8 | Exchange 15 | 9 | Exchange 15 |
| + 9 | then Press 2 | + 8 | then Press 2 |

| | | | |
|---|---|---|---|
| 8 | Press 10 then | 9 | Press 10 then |
| 8 | Press 6 | 7 | Press 6 |
| 8 | Exchange 15 | 9 | Exchange 15 |
| + 9 | then Press 2 | + 9 | then Press 3 |

Calculate by pressing the sum of two numbers at one time.

| | |
|---|---|
| 6 | Press 10 then |
| 9 | Press 5 |
| 4 | Exchange 5 then |
| 3 | Press 2 more |
| 9 | Press 20 then |
| + 9 | Clear 2 |

| | |
|---|---|
| 8 | Press 10 then |
| 4 | Press 2 |
| 9 | Press 10 then |
| 6 | Press 5 |
| 9 | Press 20 then |
| + 9 | Clear 2 |

| | |
|---|---|
| 5 | Press 8 |
| 3 | |
| 3 | Press 20 then |
| 9 | Clear 8 |
| 6 | Press 9 |
| + 3 | |

| | |
|---|---|
| 8 | Press 9 |
| 1 | |
| 6 | Press 20 then |
| 8 | Clear 6 |
| 8 | Press 10 then |
| + 5 | Press 5, Clear 2 |

| | |
|---|---|
| 5 | Press 10 then |
| 9 | Press 4 |
| 8 | Press 20 then |
| 9 | Clear 3 |
| 3 | Press 7 |
| + 4 | |

Calculate by pressing the sum of two numbers at one time.

---

| | | | |
|---|---|---|---|
| 7 | Press 9 | 8 | Press 9 |
| 2 | | 1 | |
| 6 | Press 20 then | 3 | Exchange 5 |
| 8 | Clear 6 | 2 | |
| 4 | Press 10 then | 9 | Press 20 then |
| + 9 | Press 5, Clear 2 | + 8 | Clear 3 |

---

| | | | |
|---|---|---|---|
| 9 | Press 10 then | 7 | Press 10 then |
| 9 | Press 8 | 7 | Press 4 |
| 9 | Press 20 then | 8 | Press 10 then |
| 9 | Clear 2 | 6 | Press 5, Clear 1 |
| 3 | Exchg. 5 then | 8 | Press 20 then |
| + 4 | Press 2 more | + 9 | Clear 3 |

---

| | | | |
|---|---|---|---|
| 9 | Press 10 then | 8 | Press 10 then |
| 8 | Press 7 | 7 | Press 5 |
| 1 | Press 2 | 9 | Press 10 then |
| 1 | | 5 | Press 4 |
| 6 | Press 20 then | 9 | Press 20 then |
| + 7 | Clear 7 | + 9 | Clear 2 |

---

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 9 | Press 10 then | 7 | 1 | 6 | 5 | | |
| 9 | Press 8 | 5 | 0 | 1 | 5 | | |
| 4 | Exchg. 5 then | 9 | 2 | 8 | 2 | | |
| 2 | Press 1 more | 4 | 0 | 1 | 7 | | |
| 6 | Press 10 then | 2 | 8 | 4 | 3 | | |
| + 8 | Press 5, Clear 1 | + 6 | + 0 | + 3 | + 2 | | |

---

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 7 | 8 | 9 | 6 | 2 | 7 | 7 | 2 |
| 6 | 9 | 3 | 4 | 7 | 5 | 4 | 4 |
| 9 | 4 | 8 | 2 | 1 | 3 | 6 | 5 |
| 3 | 4 | 8 | 1 | 1 | 6 | 3 | 9 |
| 2 | 3 | 1 | 3 | 4 | 9 | 1 | 8 |
| + 7 | + 6 | + 7 | + 6 | + 1 | + 0 | + 2 | + 7 |

---

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 8 | 3 | 5 | 6 | 9 | | |
| 0 | 9 | 1 | 6 | 7 | 3 | | |
| 1 | 6 | 4 | 7 | 8 | 9 | | |
| 1 | 7 | 5 | 5 | 5 | 7 | | |
| 4 | 8 | 9 | 7 | 7 | 9 | | |
| + 6 | + 7 | + 6 | + 4 | + 7 | + 9 | | |

---

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 4 | 7 | 4 | 9 | 3 | 5 | | |
| 2 | 2 | 6 | 9 | 2 | 7 | | |
| 5 | 9 | 7 | 6 | 4 | 1 | | |
| 3 | 9 | 9 | 6 | 5 | 6 | | |
| 8 | 8 | 6 | 3 | 9 | 7 | | |
| + 8 | + 6 | + 8 | + 1 | + 2 | + 6 | | |

---

| | | | |
|---|---|---|---|
| 6 | Press 10 then | | |
| 9 | Press 5 | | |
| 7 | Exchange 15 | | |
| 8 | | | |
| 7 | Press 10 then | | |
| + 7 | Press 4 | | |

---

| | | | |
|---|---|---|---|
| 8 | Press 10 then | | |
| 8 | Press 6 | | |
| 9 | Exchange 15 then | | |
| 8 | Press 2 more | | |
| 8 | Press 10 then | | |
| + 6 | Press 5, Clear 1 | | |

---

| | | | |
|---|---|---|---|
| 2 | Press 3 | | |
| 1 | | | |
| 4 | Press 10 then | | |
| 9 | Press 5, Clear 2 | | |
| 8 | Exchange 15 then | | |
| + 9 | Press 2 more | | |

---

| | | | |
|---|---|---|---|
| 8 | Press 9 | | |
| 1 | | | |
| 6 | Press 20 then | | |
| 7 | Clear 7 | | |
| 8 | Press 10 then | | |
| + 5 | Press 5, Clear 2 | | |

| | |
|---|---|
| 5 Press 10 then<br>6 Press 1 | 8 Press 10 then<br>7 Press 5 |
| 7 Press 10 then<br>7 Press 5, Clear 1 | 9 Exchg. 15 then<br>9 Press 3 more |
| 8 Exchg. 15 then<br>+ 8 Press 1 more | 7 Press 10 then<br>+ 9 Press 6 |

| | | | | |
|---|---|---|---|---|
| 3 | 8 | 9 | 6 | 9 |
| 1 | 9 | 3 | 4 | 7 |
| 1 | 4 | 9 | 9 | 9 |
| 2 | 4 | 6 | 9 | 9 |
| 6 | 3 | 8 | 7 | 6 |
| + 8 | + 6 | + 8 | + 9 | + 7 |

| | |
|---|---|
| 7 Press 10 then<br>5 Press 2 | 7 Press 10 then<br>9 Press 6 |
| 9 Press 10 then<br>4 Press 5, Clear 2 | 8 Exchg. 15 then<br>9 Press 2 more |
| 9<br>+ 6 Exchange 15 | 4 Press 10 then<br>+ 8 Press 5, Clear 3 |

| | | | | |
|---|---|---|---|---|
| 3 | 8 | 7 | 3 | 7 |
| 5 | 8 | 6 | 3 | 5 |
| 1 | 9 | 4 | 4 | 8 |
| 2 | 7 | 8 | 4 | 6 |
| 1 | 7 | 9 | 9 | 7 |
| + 3 | + 6 | + 7 | + 9 | + 9 |

| | |
|---|---|
| 5 Press 10<br>5 | 3 Press 9<br>6 |
| 2 Press 9<br>7 | 9 Press 20 then<br>3 Clear 8 |
| 7 Press 20 then<br>+ 9 Clear 4 | 7 Press 10 then<br>+ 7 Press 5, Clear 1 |

| | | | | |
|---|---|---|---|---|
| 5 | 6 | 4 | 3 | 9 |
| 2 | 8 | 2 | 5 | 4 |
| 4 | 7 | 8 | 8 | 6 |
| 1 | 4 | 9 | 8 | 9 |
| 9 | 9 | 2 | 6 | 8 |
| + 8 | + 9 | + 1 | + 6 | + 8 |

| | |
|---|---|
| 7 Press 10 then<br>5 Press 2 | 2 Press 5<br>3 |
| 8 Press 10 then<br>6 Press 5, Clear 1 | 9 Exchg. 15 then<br>7 Press 1 more |
| 9 Exchg. 15 then<br>+ 8 Press 2 more | 5 Press 10 then<br>+ 9 Press 5, Clear 1 |

Transfer carry-over number directly from left hand to right hand.

| | |
|---|---|
| 7,603 | 5,737 |
| 1,296 | 8,099 |
| 8,735 | 4,312 |
| 9,924 | 6,853 |
| 4,310 | 5,972 |
| + 5,678 | + 1,466 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 7 | 5 | 4 | 5 | 2 | 8 | 1 | 9 |
| 8 | 2 | 0 | 7 | 4 | 7 | 9 | 4 |
| 6 | 7 | 2 | 3 | 8 | 3 | 2 | 7 |
| 9 | 3 | 6 | 9 | 8 | 6 | 8 | 7 |
| 3 | 8 | 8 | 5 | 9 | 5 | 7 | 8 |
| + 7 | + 9 | + 9 | + 8 | + 9 | + 7 | + 5 | + 6 |

| | |
|---|---|
| 8,516 | 2,780 |
| 2,448 | 1,436 |
| 7,100 | 4,955 |
| 3,827 | 9,612 |
| 9,639 | 5,703 |
| + 2,579 | + 1,678 |

| | | | |
|---|---|---|---|
| 2,106 | 6,975 | 9,310 | 7,527 |
| 7,854 | 3,218 | 2,789 | 8,009 |
| 3,267 | 5,316 | 5,469 | 6,431 |
| 9,908 | 1,047 | 8,315 | 5,746 |
| 5,431 | 4,289 | 2,542 | 8,215 |
| +3,259 | +9,563 | +6,707 | +9,618 |

| | | | |
|---|---|---|---|
| 9,436 | 1,864 | 2,169 | 9,506 |
| 8,521 | 7,295 | 5,936 | 4,725 |
| 7,084 | 6,301 | 4,325 | 2,133 |
| 2,799 | 5,978 | 6,634 | 8,649 |
| 1,263 | 8,753 | 2,180 | 5,274 |
| +5,150 | +2,537 | +7,587 | +2,068 |

| | | | |
|---|---|---|---|
| 5,927 | 4,096 | 6,431 | 5,139 |
| 3,064 | 9,681 | 9,579 | 5,076 |
| 4,293 | 4,324 | 3,248 | 7,347 |
| +8,178 | +8,557 | +6,105 | +8,113 |

| | | | |
|---|---|---|---|
| 4,981 | 4,326 | 4,687 | 8,784 |
| 5,627 | 5,513 | 2,305 | 6,305 |
| 1,010 | 1,023 | 6,609 | 2,193 |
| +3,992 | +4,687 | +2,598 | +7,526 |

| | | | |
|---|---|---|---|
| 9,061 | 596 | 70 | 621 |
| 6,307 | 3,876 | 9,639 | 3,347 |
| 18 | 61 | 308 | 708 |
| 545 | 102 | 6,591 | 581 |
| 263 | 587 | 13 | 45 |
| 92 | 45 | 428 | 9,071 |
| 5,042 | 3,019 | 4,031 | 2,653 |
| +3,376 | +2,480 | + 932 | + 704 |

| | |
|---|---|
| 2,593 | 4,769 |
| 1,789 | 5,436 |
| 5,630 | 2,154 |
| +4,504 | +6,315 |

| | |
|---|---|
| 7,004 | 8,237 |
| 2,169 | 2,158 |
| 5,780 | 9,067 |
| +3,596 | +3,415 |

| | |
|---|---|
| 5,098 | 962 |
| 546 | 1,488 |
| 432 | 2,350 |
| 2,089 | 557 |
| 6,957 | 641 |
| + 376 | +3,909 |

| | |
|---|---|
| 683 | 7,416 |
| 9,067 | 923 |
| 5,439 | 321 |
| 354 | 8,841 |
| 786 | 6,349 |
| +2,103 | + 505 |

| | |
|---|---|
| 810 | 604 |
| 623 | 5,789 |
| 724 | 9,632 |
| 3,669 | 106 |
| 1,555 | 438 |
| +4,708 | +7,375 |

## HOW TO INTERNALIZE THE MATHEMATICAL OPERATIONS

As explained, in the beginning of this book, the tactile sense was a valuable asset for young children or children with learning disabilities or those requiring remedial aid. A total sensory approach to the learning of arithmetic was outlined. As the child progressed through the various stages, some of the sense impressions were reduced. For instance, when a child is first learning how to count from one to ten, we stressed the importance of concentrating on raising the voice when the two difficult manipulations of from four to five and from nine to ten were performed. Later on, this stress was less important, as the child's brain and muscle memory made these manipulations easier.

In a like manner, there comes a time when the brain can perform all the required mental calculations and it becomes time to wean ourselves from the use of the fingers, especially if we wish to reach a stage of very rapid mental calculations. Once a student performs difficult fourth stage manipulations repeatedly, and without error, he can begin to give up his dependency on the tactile sense, but it should be done gradually and in stages.

The first step is to have the student abandon the use of pressing on a desk or table. It is suggested that the student raise his hands above the desk and place his hands directly facing each other. The order to press would now be done by curling the fingers; the order to clear would be performed by straightening the fingers. While the child would still have a tactile sensation, it would not be as strong as when he directly pressed his fingers upon a hard surface.

After the student has gotten use to this lessened tactile sense, he can proceed even further to reduce his need of using his fingers. Instead of curling his fingers all the way in to touch his palm, he can move them just slightly, serving as more of a recall than actually as a calculator. The brain is, at this time, actually performing the arithmetical operations. Eventually the use of the fingers will be abandoned.

The above prescription for abandoning the use of the fingers is not be rushed, especially for the child who needs to use his fingers for self-confidence. For many slow students the Chisanbop (TM) methodology presents the first secure way they have seen for finding the right answer. It would be cruelty, indeed, to destroy their new found confidence by insisting too early that they not use their fingers. Chisanbop (TM) has shown it is not only all right, but even good to use the fingers. Don't push a child away from them until he is ready.

### HYPAI Linear Multiplication—Stage 4

In the fourth stage of multiplication, we show how to find an answer to a problem with multi-digit numbers in one straight line. We first show an example where both multipliers have two digits. We go on to show an example with three-digit numbers. Using the method shown with two- and three-digit multipliers, you can expand the method to include problems with even larger factors.

### Multiplication with Two-Digit Numbers

Linear multiplication is easiest to learn if you study the illustration below. To help you understand each step, we have presented four sets of clues. We have first named the columns with their traditional verbal names. Under that you will find the visual anchor from 3rd Stage Multiplication. Under the visual anchor you will find the part of the diagram that is operational for that column. Finally you will find the partial products that must be placed into that column.

| | | 4o<br>3o | 7<br>5 | |
|---|---|---|---|---|
| x | | | | |
| **Thousands** | **Hundreds** | **Tens** | **Units** | **Verbal Anchor** |
| | oo | o | | **Visual Anchor** |
| | ③ | ② | ① | **Diagram** |
| | 3o x 4o | 5 x 4o<br>3o x 7 | 5 x 7 | **Partial Product** |
| 1 | 6 | 4 | 5 | **Column Answer** |

As in 3rd Stage Multiplication, the columns identify where the number on the right hand should be placed. Also, as in 3rd Stage Multiplication, it is easiest to identify the columns by their visual anchor.

The first multiplication to be performed is 5 x 7. Since there is no implicit zero, a 5 is written into the units' column. This is followed by transferring the 3 to the right hand.

The next step is to add the products of 5 x 4o and 3o x 7. Thus you first press a 20 and then another 21. Your hands should be reading 44. Since both partial products contained a factor with one implicit zero, the 4 on the right hand is placed into the column with one zero. You transfer the 4 on the left hand to the right hand.

The last step is to multiply 3o x 4o. You add 12 to your fingers and your hands should be reading 16. The six on the right hand belongs to the column with *two* zeroes since *each* of the factors contained one implicit zero. You therefore place a 6 into the hundreds' column. The carry-over on the left hand, 1, automatically goes into the next column on the left, the thousands' column. Your answer is now complete.

Each page of problems contains diagrams to help you remember which factors go into which column. Study the diagrams carefully before attempting a problem. If it is helpful, you may want to write the visual anchors down before attempting the problem.

## Carry-over Numbers Involving Two Digits

On some problems, you will run into carry-over numbers with two digits. For instance, in the problem 89 x 67, you start the second step with a carry-over of 6. To this must be added the products of 9 x 6o and 8o x 7. You run out of fingers, but there is an easy solution. When the left hand is filled and you want to

add another ten simply clear the left hand and touch it to your forehead. You are thereby storing a hundred. However, you can now no longer read your fingers, but must read your answer. The answer is 116. You write the 6 into the column with one zero. Now when you transfer, you must transfer a 1 from the left hand to the right *and* another 1 from the brain to the left hand. To make the latter move more memorable, once again touch your forehead.

## EXERCISES

| | | | |
|---|---|---|---|
| 40 x 56 | 72 x 35 | 93 x 11 | Diagram ③ ② ① |
| 74 x 25 | 86 x 36 | 43 x 64 | |
| 71 x 52 | 63 x 26 | 84 x 32 | Diagram ③ ② ① |
| 53 x 46 | 14 x 35 | 65 x 47 | |
| 65 x 56 | 42 x 46 | 29 x 43 | Diagram ③ ② ① |
| 73 x 70 | 39 x 57 | 82 x 18 | |
| 82 x 19 | 91 x 82 | 42 x 53 | Diagram ③ ② ① |
| 56 x 53 | 42 x 24 | 83 x 49 | |
| 24 x 93 | 36 x 32 | 27 x 31 | Diagram ③ ② ① |
| 63 x 32 | 18 x 74 | 50 x 67 | |

| | | | |
|---|---|---|---|
| 77 x 44 | 55 x 66 | 88 x 22 | Diagram ③ ② ① |
| 99 x 11 | 30 x 19 | 67 x 76 | |
| 46 x 28 | 63 x 46 | | Diagram ③ ② ① |
| 95 x 83 | 18 x 68 | | |
| 75 x 46 | 98 x 73 | | Diagram ③ ② ① |
| 86 x 18 | 33 x 33 | | |
| 30 x 61 | 71 x 12 | | Diagram ③ ② ① |
| 36 x 80 | 55 x 66 | | |

Carry-over numbers involving two digits

| | | |
|---|---|---|
| 87 x 59 | 69 x 86 | Diagram ③ ② ① |
| 98 x 86 | 79 x 95 | |

## Multiplication with Three-Digit Numbers

The following example illustrates how to perform straight line multiplication with three-digit factors. Once again, by using the visual anchor you should have no problem in knowing where to place the partial products.

| | | | | 3oo | 7o | 6 |
|---|---|---|---|---|---|---|
| x | | | | 4oo | 2o | 8 |

| Hundred Thousands | Ten Thousands | Thousands | Hundreds | Tens | Units | Verbal Anchor |
|---|---|---|---|---|---|---|
| | ooo<br>o | ooo | oo | o | | Visual Anchor |
| | ⑤ \| | ④ X | ③ X \| | ② X | ① \| | Diagram |
| | 4oo x 3oo | 2o x 3oo<br>4oo x 7o | 8 x 3oo<br>2o x 7o<br>4oo x 6 | 8 x 7o<br>2o x 6 | 8 x 6 | Partial Product |
| 1 | 6 | 0 | 9 | 2 | 8 | Column Answer |

In the above example, the first step is to multiply 8 x 6. The answer is 48. The 8 should be placed in the units' column since there is no implicit zero involved. The 4 should be transferred. To this 4 you must add the products of 8 x 7o and 2o x 6. You therefore press 56 and then 12. Your hands should be reading 72. You know that the 2 registered on the right hand should be placed into the column with one zero since each of the partial products had a factor with one implicit zero. After you have transferred the 7, you must add the products of 8 x 3oo, 2o x 7o, and 4oo x 6. Your fingers should be reading 69. Since all the partial products have two implicit zeroes, the 9 on the right hand must be placed into the column with two zeroes.

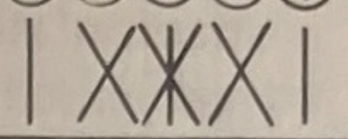
After you have transferred the 6, you must add the products of 2o x 3oo and 4oo x 7o. After making the additions, your fingers should read 40. Since the right hand has nothing registered on it, a zero must be placed into the column with three zeroes. This is because both the partial products had factors with three implicit zeroes.

After the 4 is transferred, you add your last partial product, 4oo x 3oo. Your fingers should be registering 16. The 6 of the right hand is written into the column with 4 zeroes because the factors of the product had four implicit zeroes. The 1 on the left hand is written into the next column since this was a carry-over number.

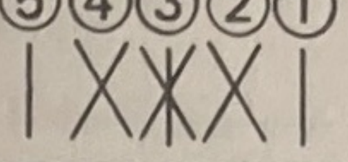
The same format can be used when one of the multipliers only has two digits. Simply write a zero in to the hundreds' place and act as if it has three digits.

## EXERCISES

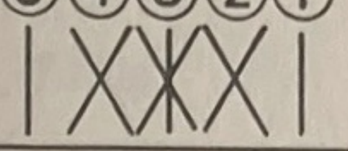
721 x 435 | 902 x 436 | 275 x 564 | Diagram ⑤④③②①



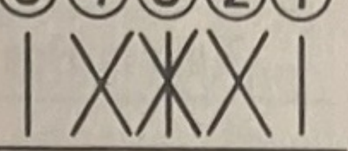
613 x 820 | 736 x 712 | 643 x 856 | Diagram ⑤④③②①



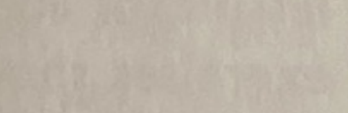
115 x 239 | 678 x 345 | 217 x 721 | Diagram ⑤④③②①



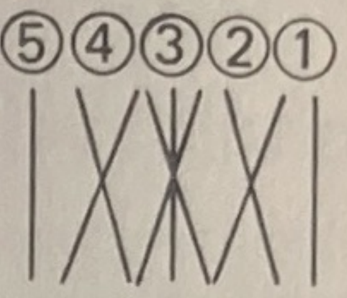
903 x 907 | 864 x 105 | 808 x 803 | Diagram ⑤④③②①



126 x 620 | 416 x 278 | 327 x 987 | Diagram ⑤④③②①

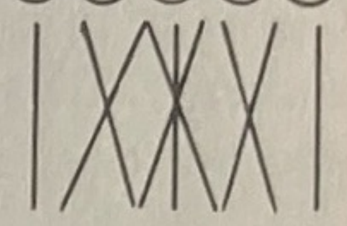


402 x ○45 | 715 x ○64 | Diagram ⑤④③②①



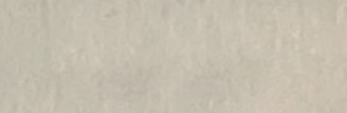
376 x ○95 | 743 x ○52

632 x ○42 | 183 x ○71 | Diagram ⑤④③②①



71 x 461 | 38 x 243

○29 x 343 | Diagram ⑤④③②①



○78 x 123 | ○60 x 904

## SUBTRACTION STAGE 1

### Part (A)

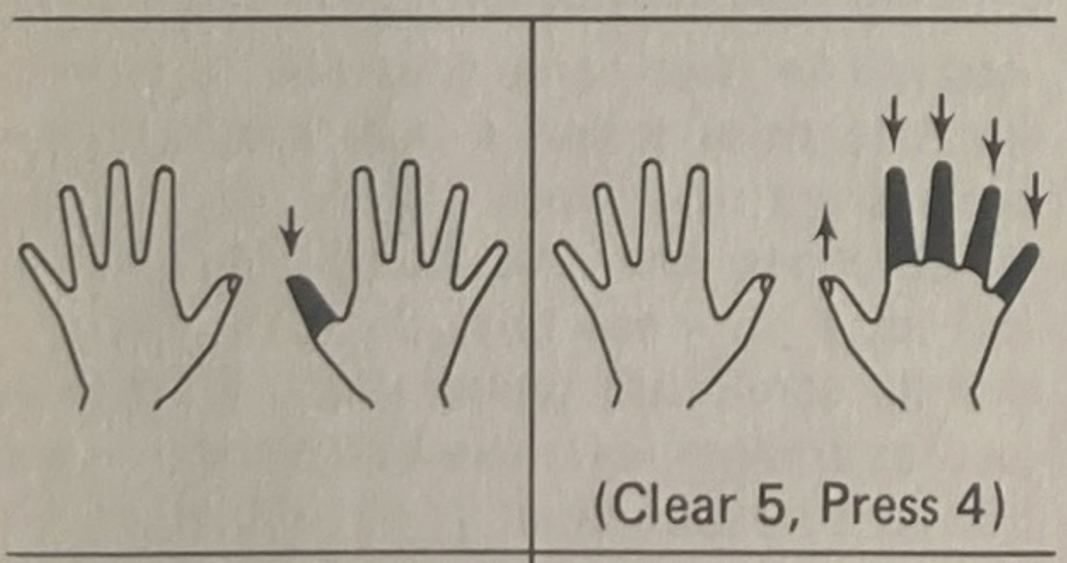
## DAILY ORAL PRACTICE

Beginning work on Chisanbop ™ Subtraction requires a considerable amount of practice in clearing the fingers one-by-one. It is best to start with numbers less than 5 and then clear the fingers one-by-one until 0 is reached. For example, Press 4 and then clear the fingers one-by-one, and say "Clear one, and another one, and another one, etc."

The two key manipulations in Stage 1 Subtraction are those required when the numbers 5 or 10 are pressed and it is necessary to clear more fingers (Examples 5–1 or 10–1). The terms below will be used throughout this book to identify these two important manipulations.

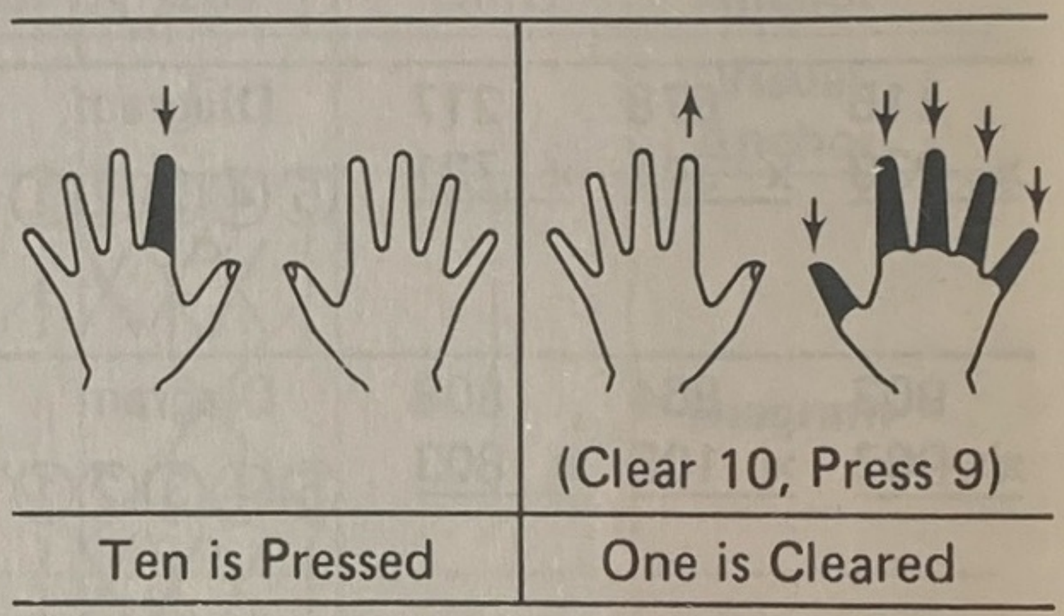
**(1) 5 – 1**
**Clear 5, Press 4**

The term Clear 5, Press 4 refers to the manipulation of clearing the thumb on the Right Hand as the four fingers of the same hand are pressed simultaneously. See diagram.




**(2) 10 – 1**
**Clear 10, Press 9**

The term Clear 10, Press 9 refers to the manipulation of clearing one of the ten-valued fingers of the Left Hand as all of the fingers of the Right Hand are pressed simultaneously. See diagram.




## DAILY PRACTICE EXERCISES

In the same manner that counting exercises were practiced before each day's lesson for addition, the following types of exercises should now be used for subtraction. Each exercise requires an initial press of a tens' number which is then reduced to zero with unit-by-unit, Stage 1 Subtraction. The two manipulations that will require a great deal of attention by both children and adults are the ones where movement is made from the Left Hand to the Right Hand (Clear 10, Press 9) and where movement is made from the Right Thumb to the other four fingers on this hand (Clear 5, Press 4). Examples such as 5–1, 10–1, 20–1, 30–1, etc., should be emphasized and practiced daily.

A. PRESS 10. SUBTRACT ONES, in sequence, saying: clear one and another one and another one....etc., 10 times until zero is reached.

B. PRESS 20. SUBTRACT TWOS, by Units. Clear two saying: *One, two,* and another two saying *one, two*...etc., 10 times, until zero is reached.

C. PRESS 30. SUBTRACT THREES, by Units. Clear three saying: *one, two, three,* and another three saying: *one, two, three,* and another three saying: one, two, three, etc. 10 times, until zero is reached.

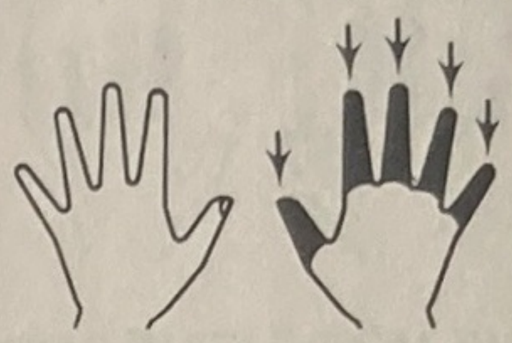
D. PRESS 40. SUBTRACT FOURS, by Units. Clear four saying: *One, two, three, four,* and another four saying: *one, two, three, four,* and another four saying: *one, two, three, four,* etc. 10 times, until zero is reached.

E. PRESS 50. SUBTRACT FIVES, by Units. Clear five saying: *one, two, three, four, five,* and another five saying: *one, two, three, four, five,* and another five saying: *one, two, three, four, five,* etc., 10 times until zero is reached.

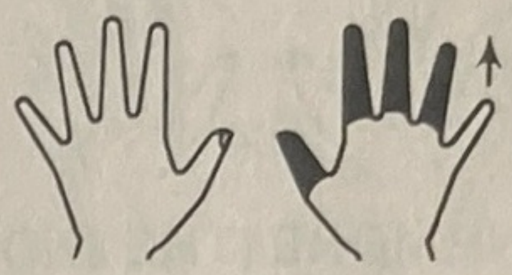
The following pages present diagrams which illustrate correct manipulations and oral expressions to use in oral practice in clearing the fingers. Whenever the students start to do the actual subtraction problems, they should count as they clear their fingers. Important: The fingers of each hand are cleared from the outside fingers to the inside fingers. For example, if all of the fingers of the right hand were pressed, the pinky finger would be cleared first, then the ring finger, the middle finger, the index finger, in that order, as necessary. This is the opposite or reverse of the addition process because when numbers are pressed in addition, we always begin with the index finger and then move to the outside fingers as necessary.

## ORDER OF CLEARING THE FINGERS ON THE RIGHT HAND

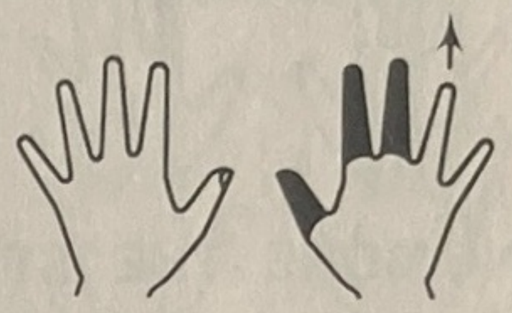
*PRESS* NINE DIRECTLY OR ONE-BY-ONE



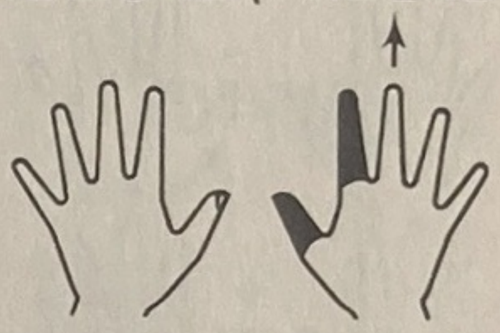
*CLEAR* ONE (THE 4TH FINGER)



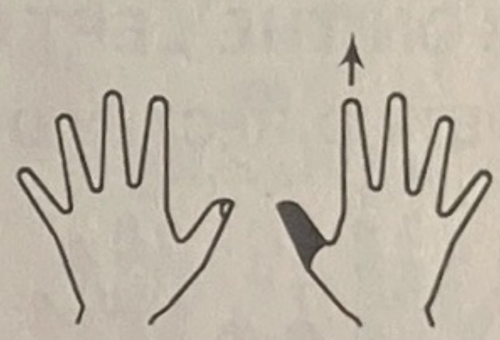
*CLEAR* ONE MORE (THE 3RD FINGER)



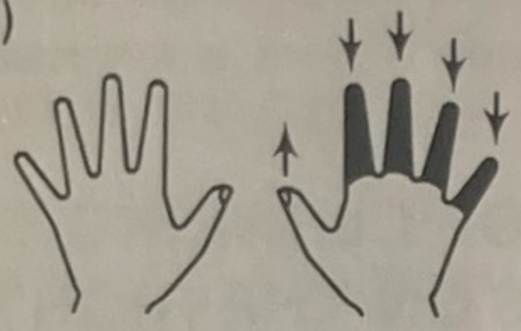
*CLEAR* ONE MORE (THE 2ND FINGER)



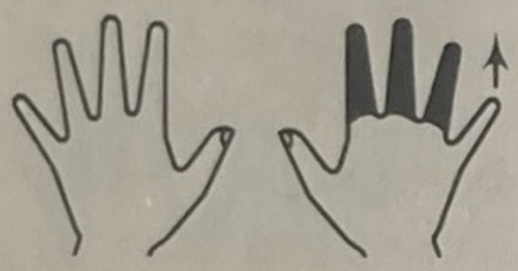
*CLEAR* ONE MORE (THE 1ST FINGER)



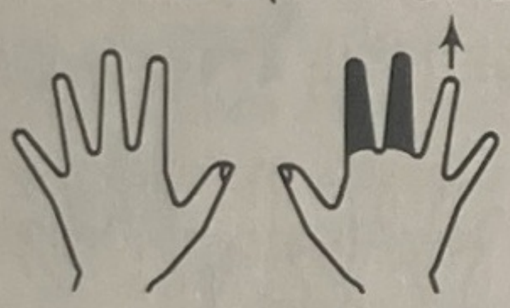
*CLEAR* ONE MORE (CLEAR THE 5TH FINGER AND SIMULTANEOUSLY PRESS THE 1ST, 2ND, 3RD, AND 4TH FINGERS)



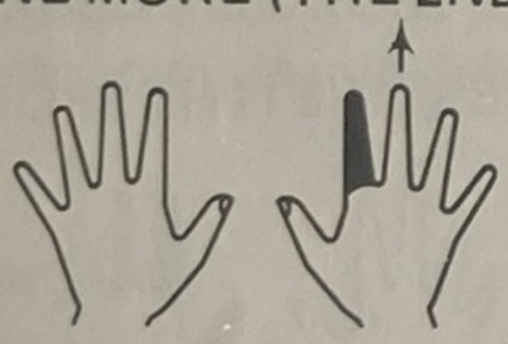
*CLEAR* ONE MORE (THE 4TH FINGER)



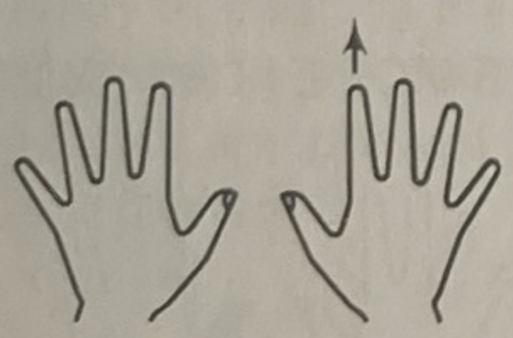
*CLEAR* ONE MORE (THE 3RD FINGER)



*CLEAR* ONE MORE (THE 2ND FINGER)



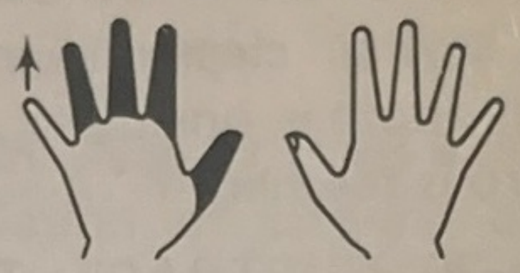
*CLEAR* ONE MORE (THE 1ST FINGER)



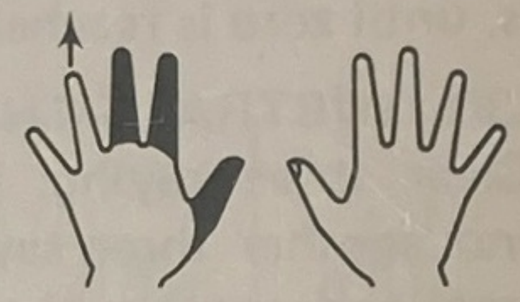
## ORDER OF CLEARING THE FINGERS ON THE LEFT HAND

*PRESS* NINETY DIRECTLY OR TEN-BY-TEN

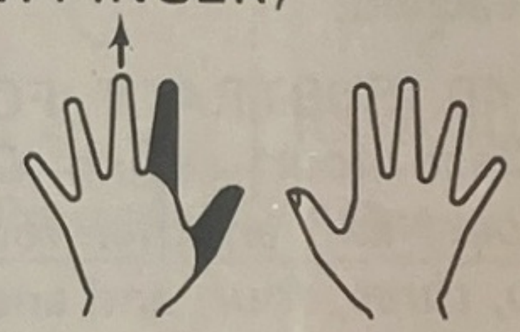
*CLEAR* TEN (THE 40TH FINGER)



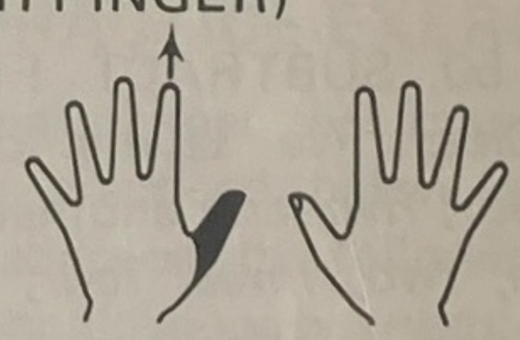
*CLEAR* ANOTHER TEN (THE 30TH FINGER)



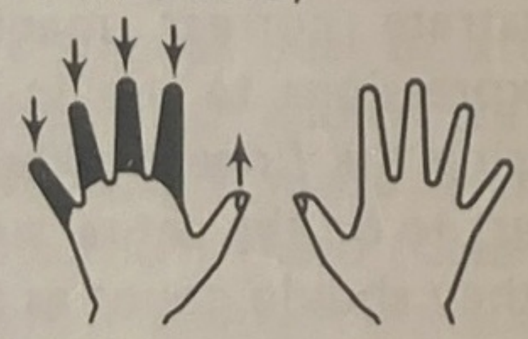
*CLEAR* ANOTHER TEN (THE 20TH FINGER)



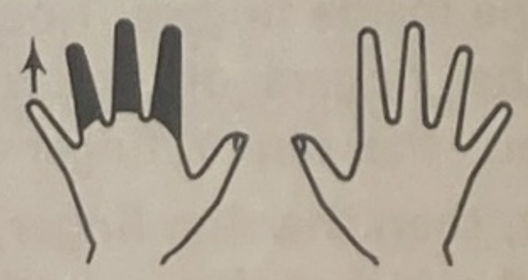
*CLEAR* ANOTHER TEN (THE 10TH FINGER)



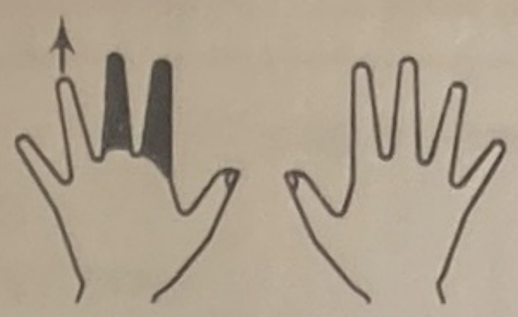
*CLEAR* ANOTHER TEN (CLEAR THE 50TH FINGER AND SIMULTANEOUSLY PRESS THE 10TH, 20TH, 30TH, AND 40TH FINGERS)



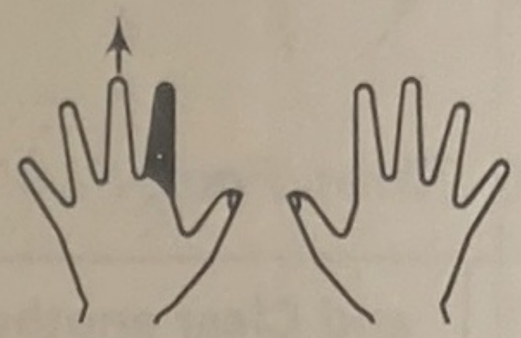
*CLEAR* ANOTHER TEN (THE 40TH FINGER)



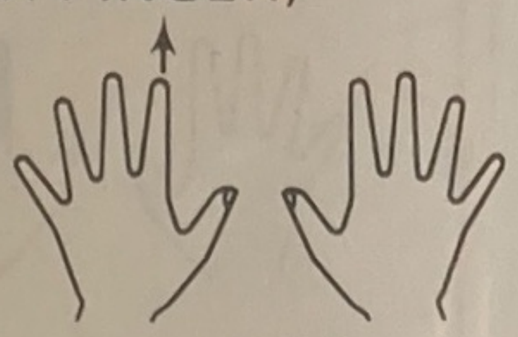
*CLEAR* ANOTHER TEN
(THE 30TH FINGER)



*CLEAR* ANOTHER TEN
(THE 20TH FINGER)



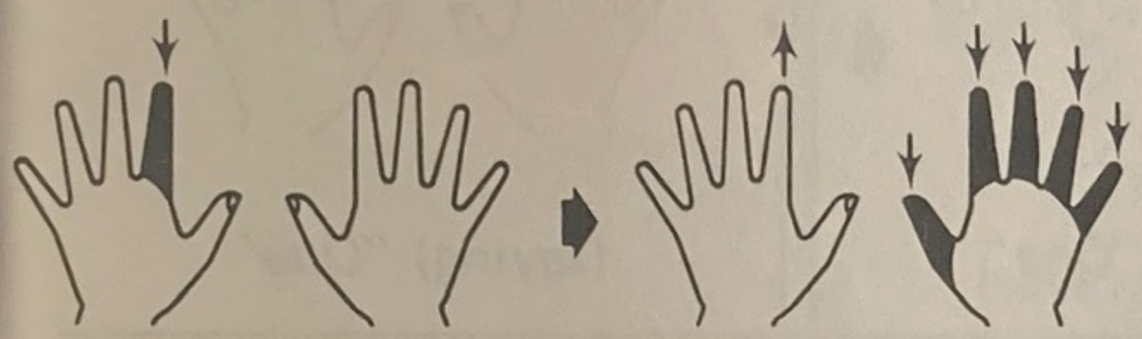
*CLEAR* ANOTHER TEN
(THE 10TH FINGER)



## CLEAR THE FINGERS USING BOTH HANDS

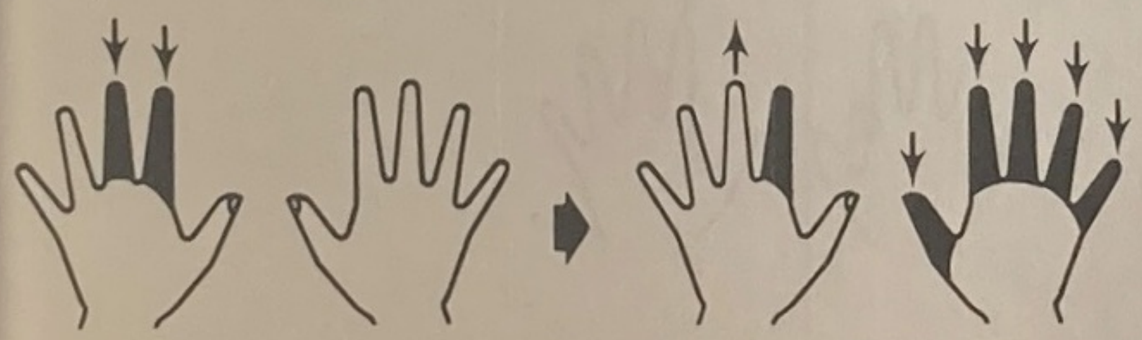
**10 - 1**

PRESS 10 CLEAR 1



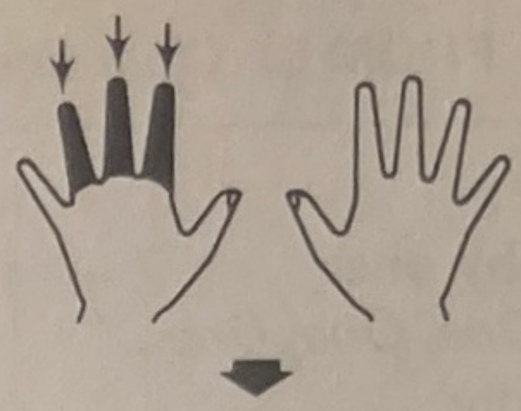
**20 - 1**

PRESS 20 CLEAR 1

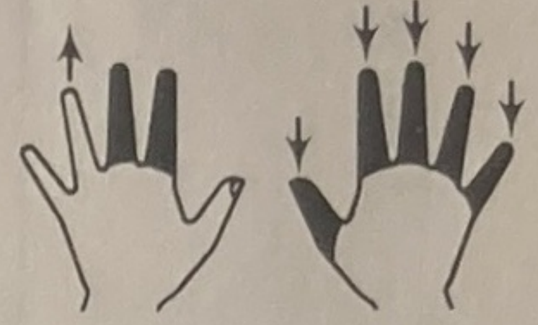


**30 - 1**

PRESS 30

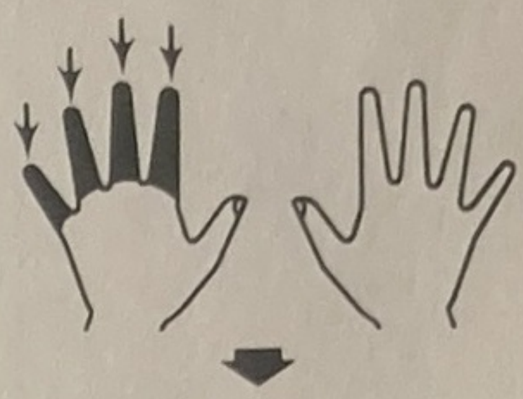


CLEAR 1

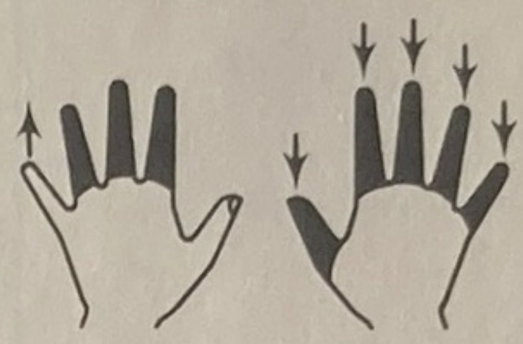


**40 - 1**

PRESS 40

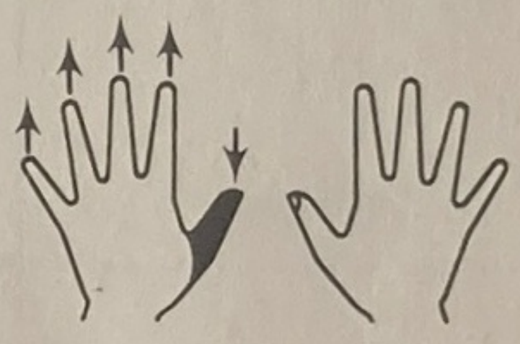


CLEAR 1

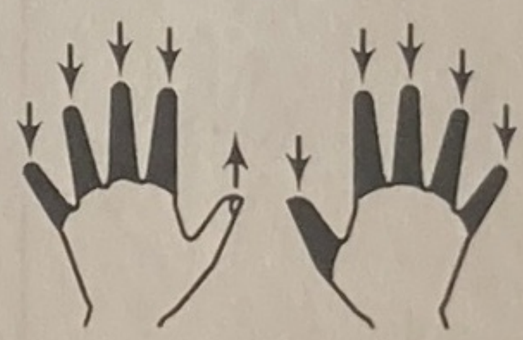


**50 - 1**

PRESS 50



CLEAR 1



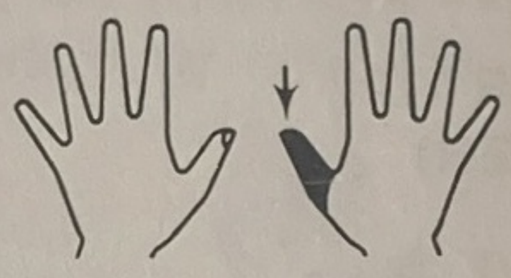
## ORAL PRACTICE

**(CLEAR 5 FROM 5)** Clear one-by-one (saying) *"One, One, One..."* until zero is reached

EXAMPLE:
**Clearing 5 by ones**
(saying) *"One, One, One..."*
repeatedly

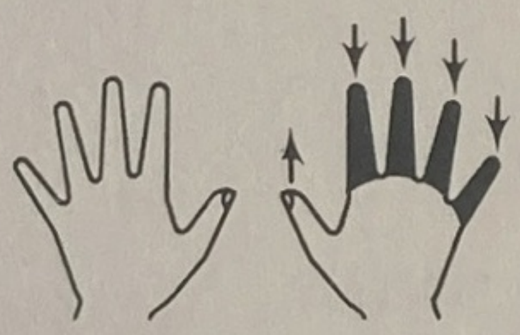
**(A)** 5
**(B)** − 5
□

**(A) Press 5 (directly or) one-by-one**



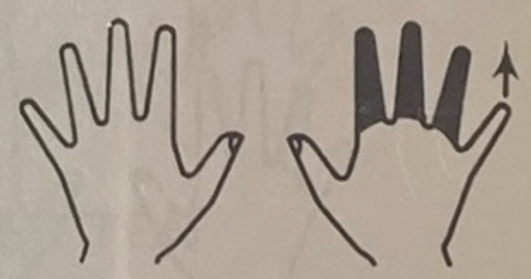
(saying) *"One, Two, Three, Four, Five"*

**(B) Clear 1**



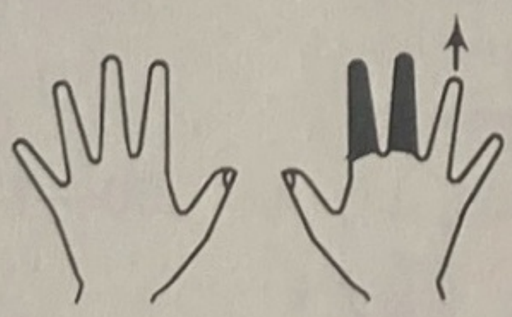
(saying) *"One"*
Through: (Clear 5, Press 4)

**and Clear another 1**



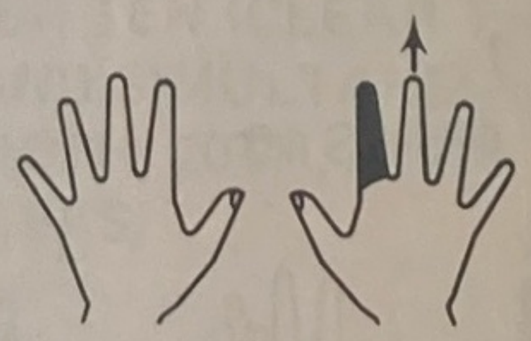
(saying) *"One"*

**and Clear another 1**



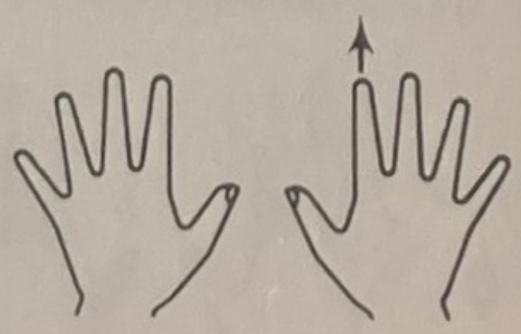
(saying) *"One"*

**and Clear another 1**



(saying) *"One"*

**and Clear another 1**



(saying) *"One"*

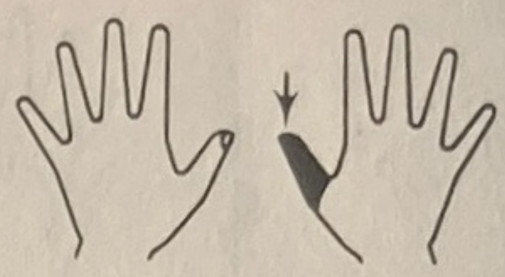
## ORAL PRACTICE

**(CLEAR 5 FROM 5)** Clear one-by-one (saying) *"One, Two, Three, Four, Five"* until zero is reached

EXAMPLE:
**Clearing 5 by ones**
(saying) *"One, Two, Three, Four, Five"*

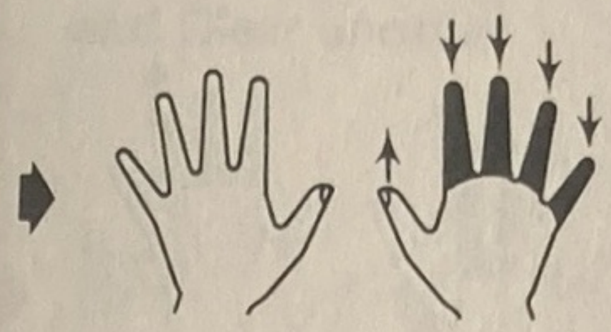
**(A)** 5
**(B)** − 5
□

**(A) Press 5 (directly or) one-by-one**



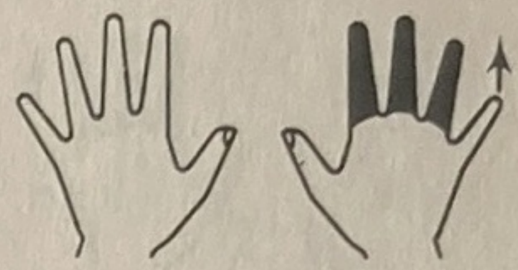
(saying) *"One, Two, Three, Four, Five"*

**(B) Clear 1**



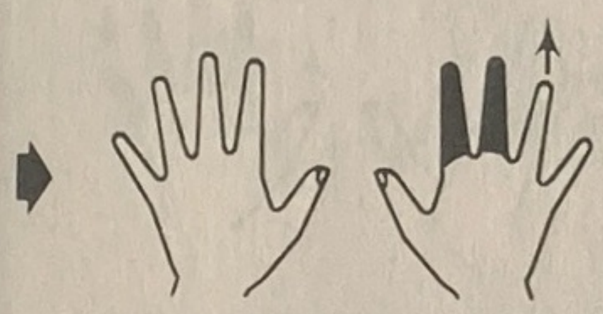
(saying) *"One"*
Through: (Clear 5, Press 4)

**and Clear another 1**



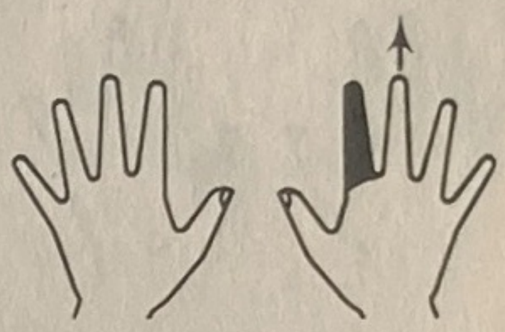
(saying) *"Two"*

**and Clear another 1**



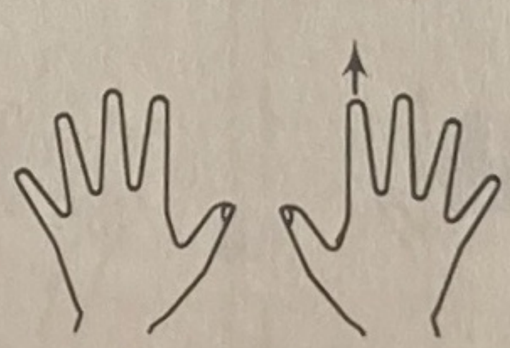
(saying) *"Three"*

**and Clear another 1**



(saying) *"Four"*

**and Clear another 1**



(saying) *"Five"*

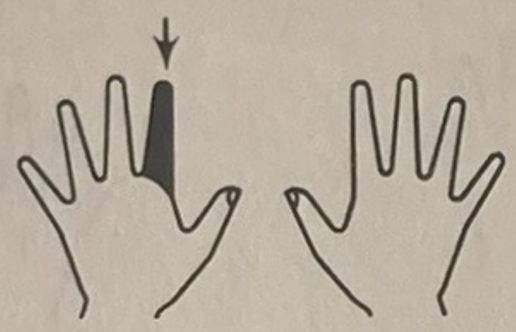
## ORAL PRACTICE

**(CLEAR 10 FROM 10)** Clear one-by-one (saying) *"One, One, One..."* until zero is reached

EXAMPLE:
**Clearing 10 by ones**
(saying) *"One, One, One..."*
repeatedly

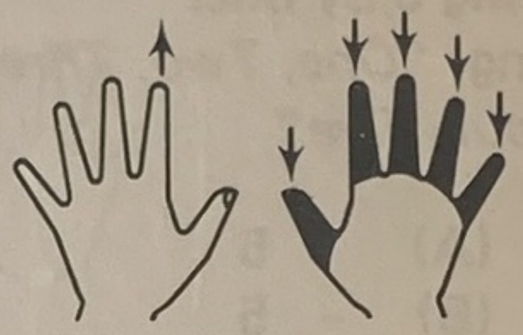
**(A)** 10
**(B)** − 10
□

**(A) Press 10 directly (or one-by-one)**



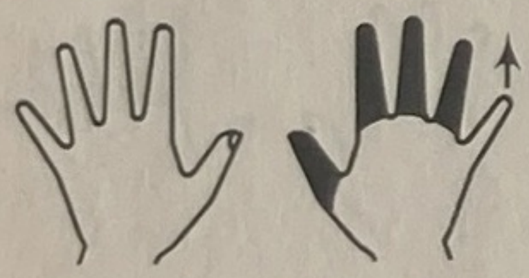
(saying) *"Onety"*

**(B) Clear 1**



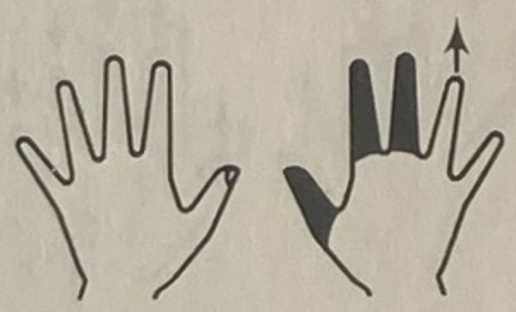
(saying) *"One"*
Through: (Clear 10, Press 9)

**and Clear another 1**



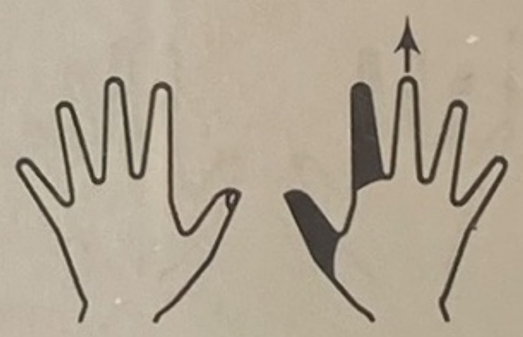
(saying) *"One"*

**and Clear another 1**



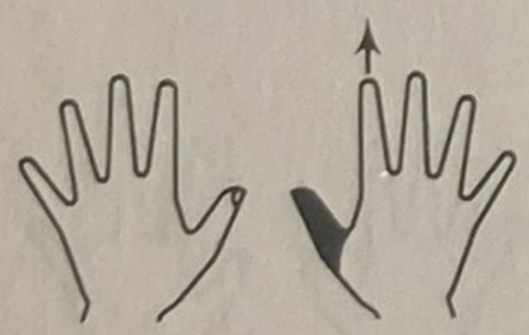
(saying) *"One"*

**and Clear another 1**



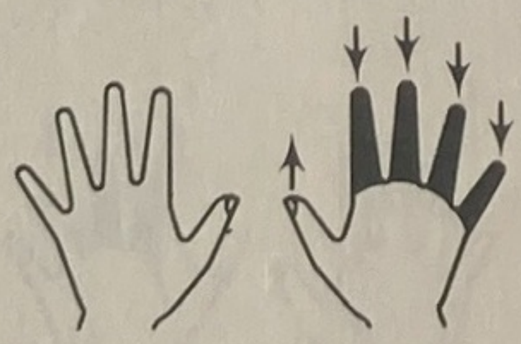
(saying) *"One"*

**and Clear another 1**



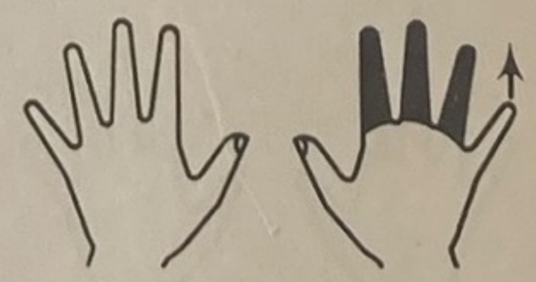
(saying) *"One"*

**and Clear another 1**



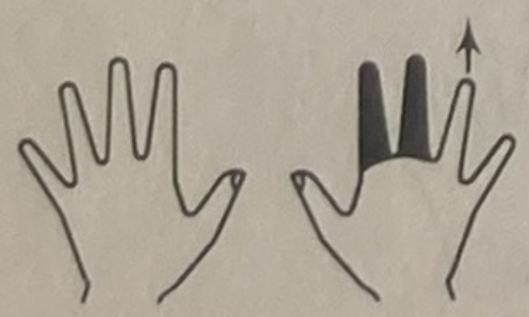
(saying) *"One"*
Through: (Clear 5, Press 4)

**and Clear another 1**



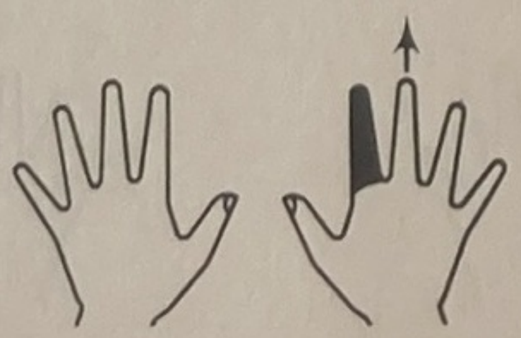
(saying) *"One"*

**and Clear another 1**



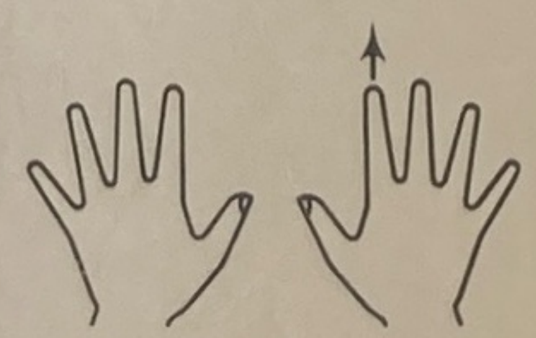
(saying) *"One"*

**and Clear another 1**



(saying) *"One"*

**and Clear another 1**



(saying) *"One"*

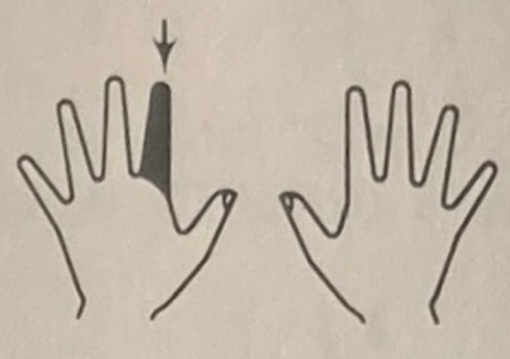
## ORAL PRACTICE

**(CLEAR 10 FROM 10)** Clear one-by-one (saying) *"One, Two, Three, Four..."* until zero is reached

EXAMPLE:
**Clearing 10 by ones**
(saying) *"One, Two, Three, Four, Five, Six, Seven, Eight, Nine, Ten"*

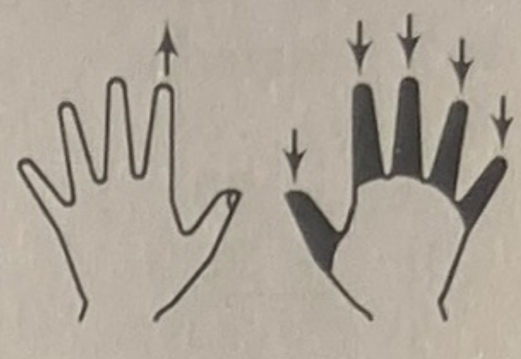
**(A)** 10
**(B)** − 10
□

**(A) Press 10 directly (or one-by-one)**



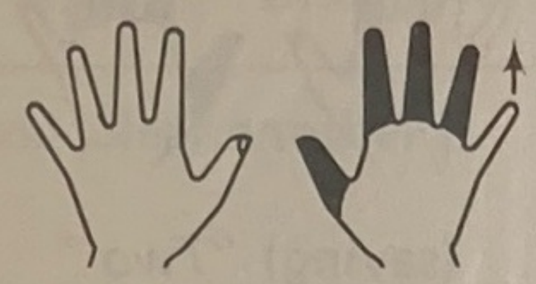
(saying) *"Onety"*

**(B) Clear 1**



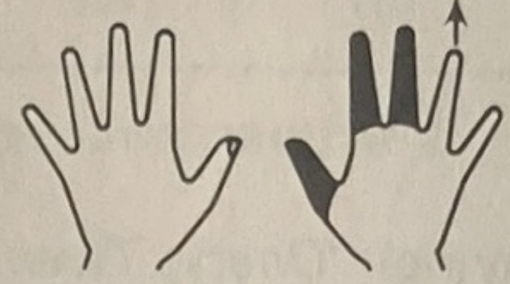
(saying) *"One"*
Through: (Clear 10, Press 9)

**and Clear another 1**



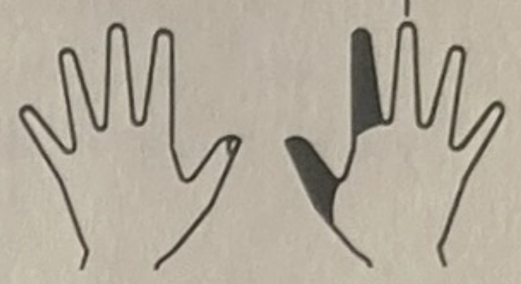
(saying) *"Two"*

**and Clear another 1**



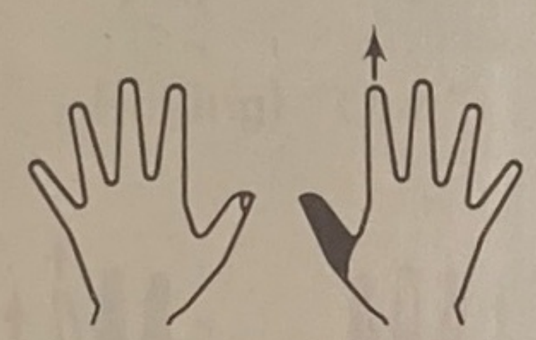
(saying) *"Three"*

**and Clear another 1**



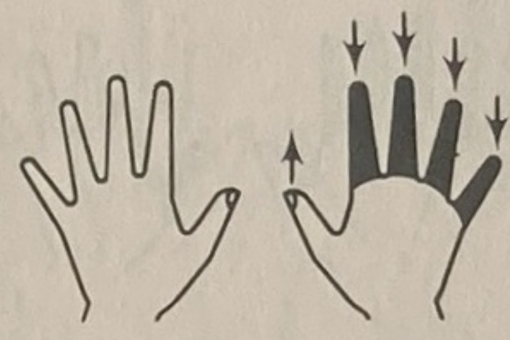
(saying) *"Four"*

**and Clear another 1**



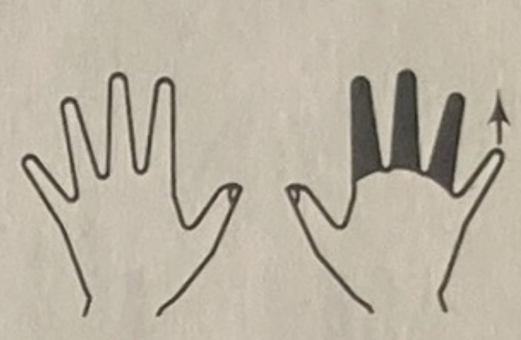
(saying) *"Five"*

**and Clear another 1**



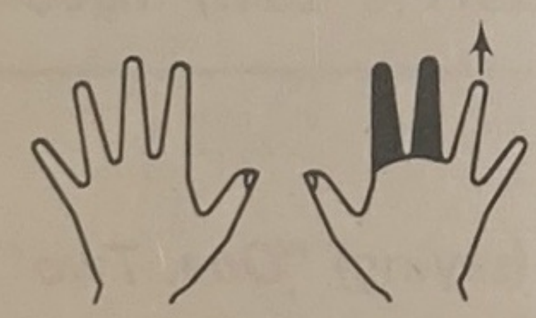
(saying) *"Six"*
Through: (Clear 5, Press 4)

**and Clear another 1**



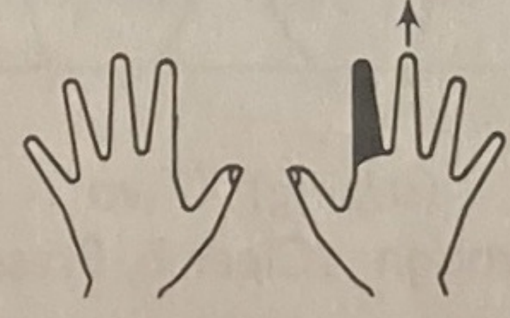
(saying) *"Seven"*

**and Clear another 1**



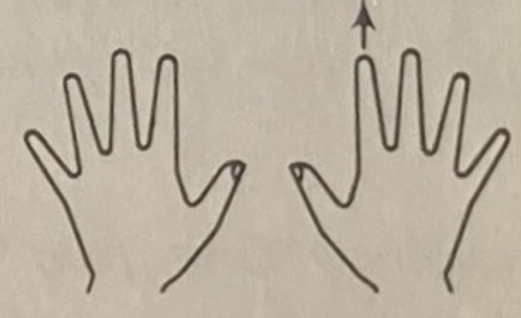
(saying) *"Eight"*

**and Clear another 1**



(saying) *"Nine"*

**and Clear another 1**



(saying) *"Ten"*

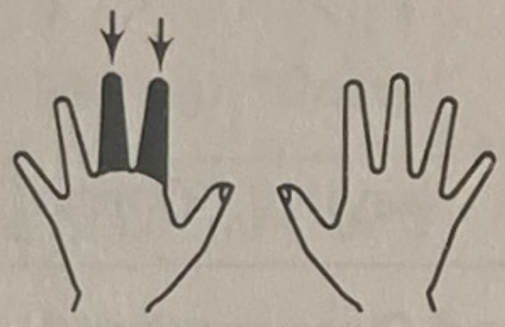
## ORAL PRACTICE

**(CLEAR 20 FROM 20)** Clear twos one-by-one (saying) *"One, Two, One, Two..."* until zero is reached

EXAMPLE:
**Clearing 20 by twos**
(saying) *"One, Two"*... *"One, Two"* repeatedly

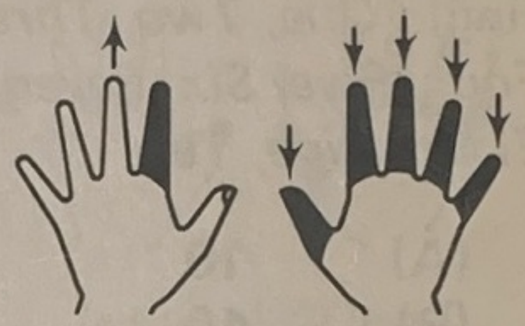
**(A) 20**
**(B) - 20**
**□**

**(A) Press 20 ten-by-ten (or one-by-one)**

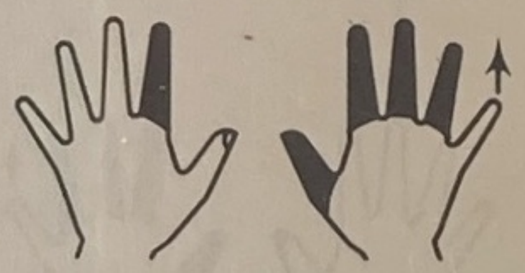


(saying) *"Onety, Twenty"*

**(B) Clear 2**

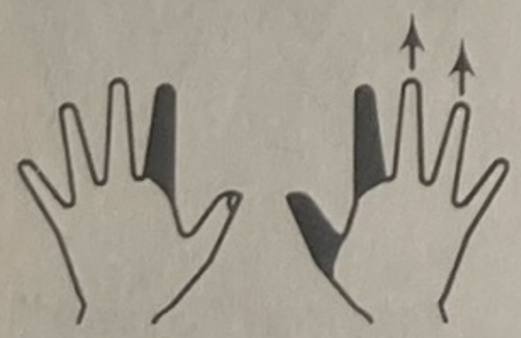


(saying) *"One"*
Through: (Clear 10, Press 9)



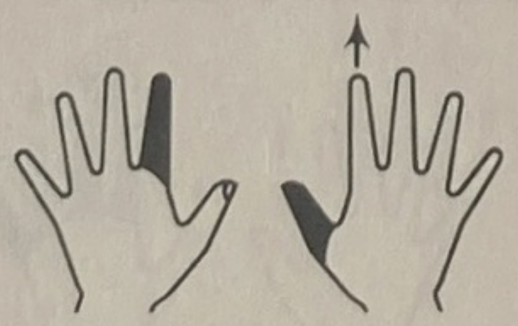
(saying) *"Two"*

**and Clear another 2**

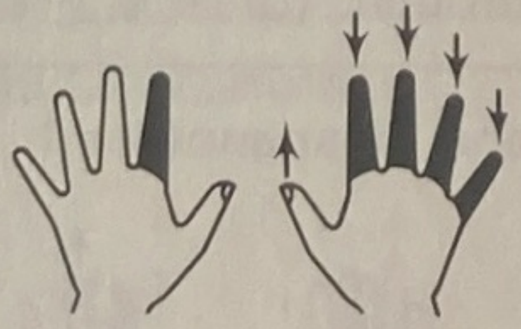


(saying) *"One, Two"*

**and Clear another 2**

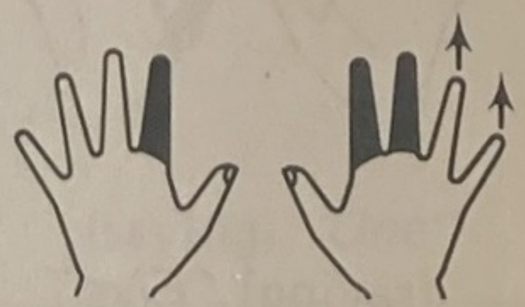


(saying) *"One"*

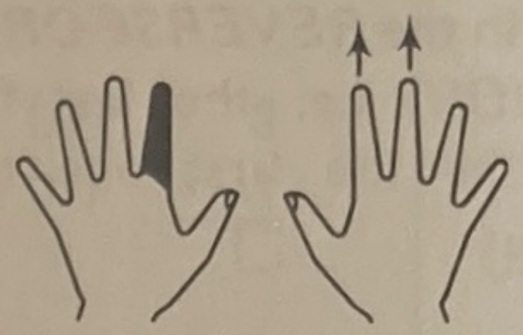
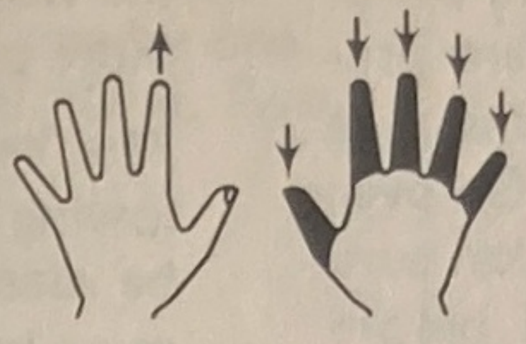
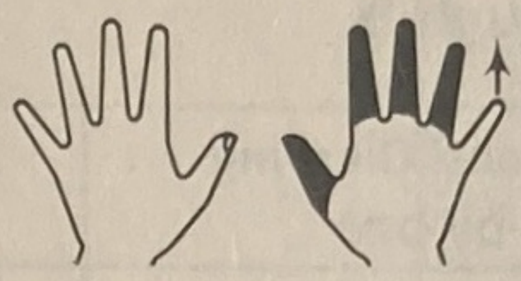
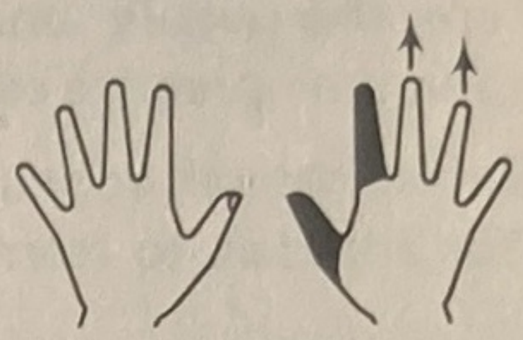
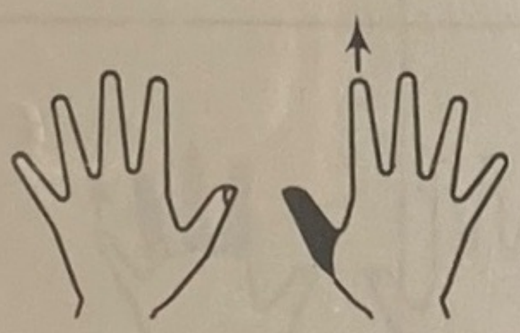
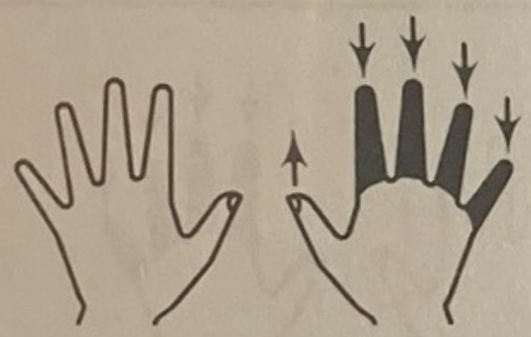
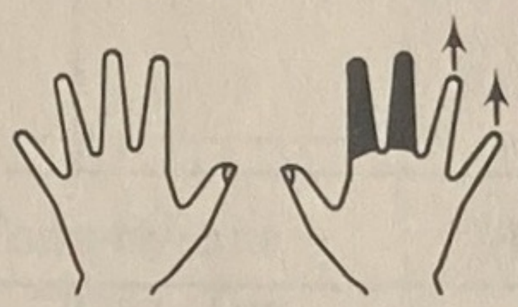
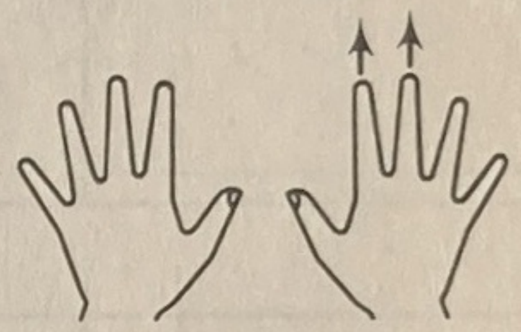


(saying) *"Two"*
Through: (Clear 5, Press 4)

**and Clear another 2**



(saying) *"One, Two"*

| | | |
|---|---|---|
| **and Clear another 2** | **and Clear another 2** | **and Clear another 2** |
|  <br>(saying) *"One, Two"* |  <br>(saying) *"One"*<br>Through: (Clear 10, Press 9)<br>  <br>(saying) *"Two"* |  <br>(saying) *"One, Two"* |
| **and Clear another 2** | **and Clear another 2** | **and Clear another 2** |
|  <br>(saying) *"One"*<br>  <br>(saying) *"Two"*<br>Through: (Clear 5, Press 4) |  <br>(saying) *"One, Two"* |  <br>(saying) *"One, Two"* |

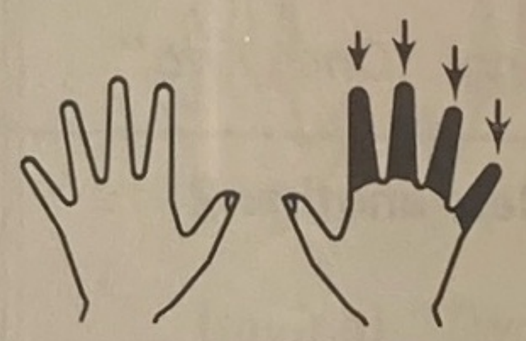
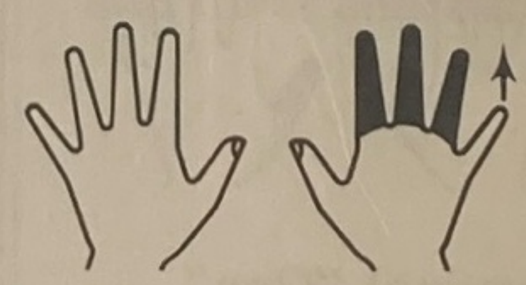
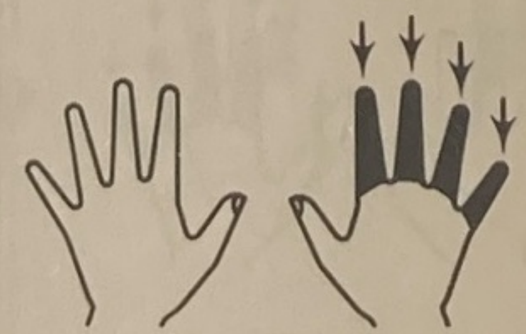
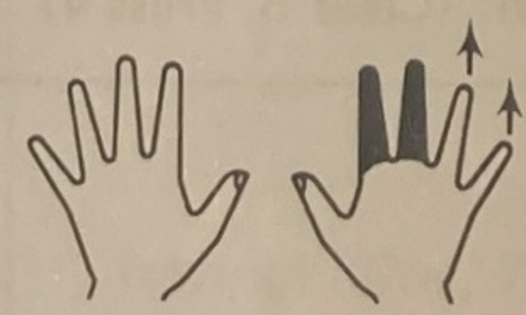
## Practice Exercises

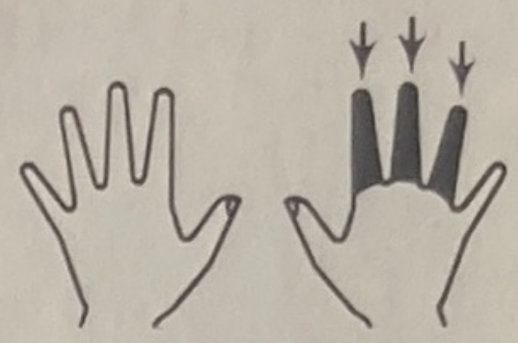
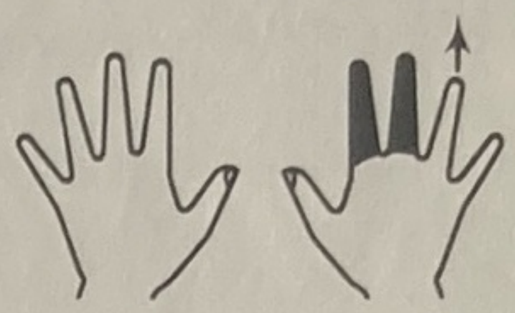
The following sections of this book present different types of subtraction problems. For each type, examples are presented followed by many problems that can be used to provide practice. Go over the example problems and make sure they are thoroughly understood before attempting the practice exercises.

While this manual contains enough exercises for an adult to learn Chisanbop ™ Subtraction, children should make use of the workbooks where they will find many more practice problems.

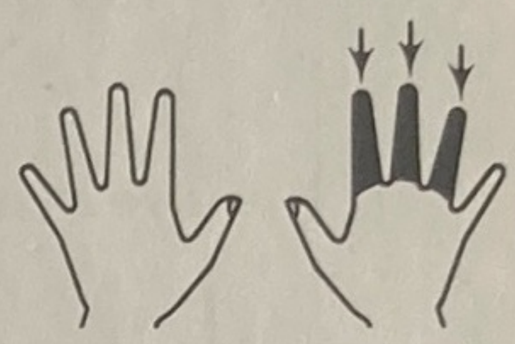
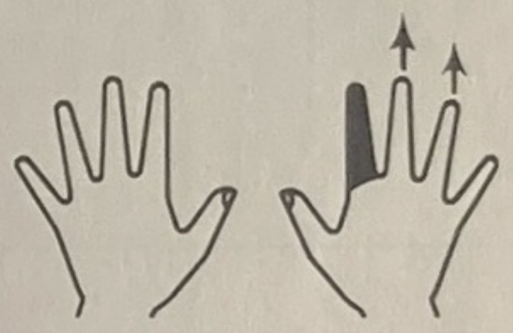
The key concepts to focus on in the following problems are: 1) the fingers should be cleared *ONE-BY-ONE;* and 2) fingers must be cleared in the *REVERSE ORDER FROM ADDITION,* i.e. the last finger pressed should be the first one to be cleared.

## Right Hand—Numbers 0 through 5

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 4<br>(B) − 1<br>□ | **(A) Press 4 one-by-one**<br>(saying) *"One, Two, Three, Four"* | **(A)**  |
| | **(B) Clear 1**<br>(saying) *"One"* | **(B)**  |
| | | *"Read Your Fingers"* **Answer:** *"3"* |
| (A) 4<br>(B) − 2<br>□ | **(A) Press 4 one-by-one**<br>(saying) *"One, Two, Three, Four"* | **(A)**  |
| | **(B) Clear 2 one-by-one**<br>(saying) *"One, Two"* | **(B)**  |
| | | *"Read Your Fingers"* **Answer:** *"2"* |

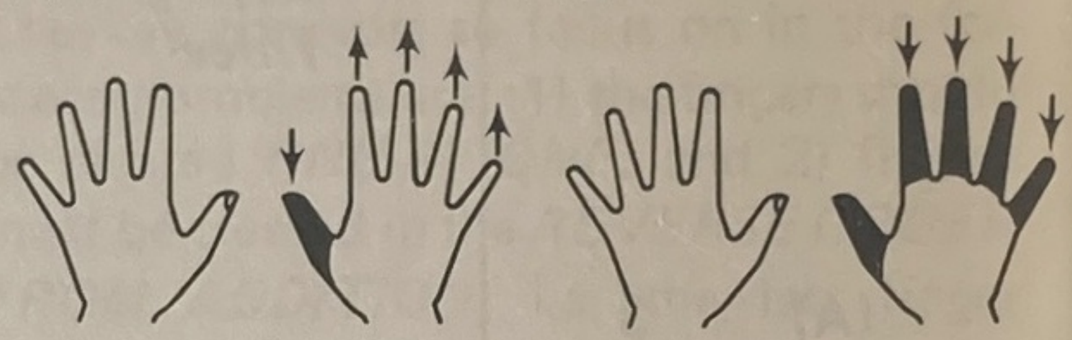
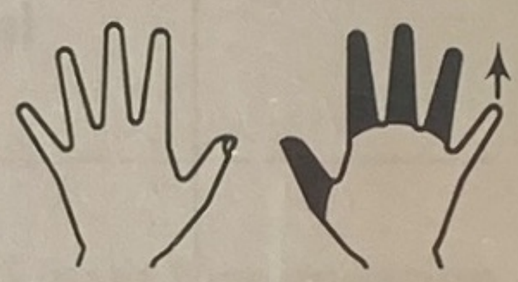
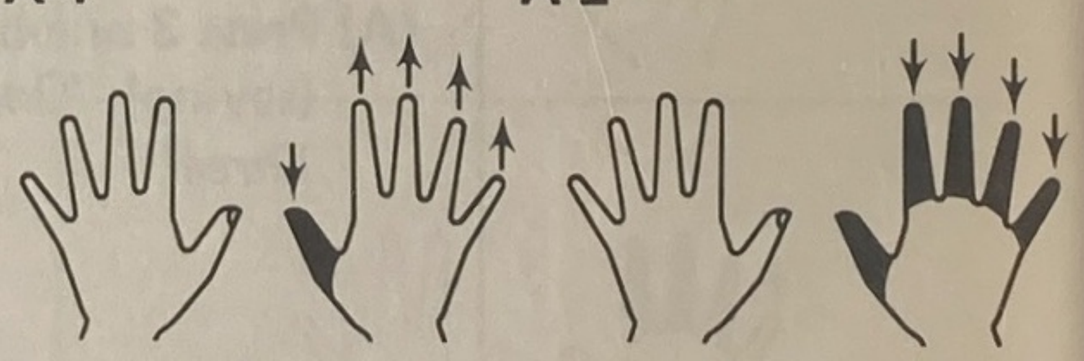
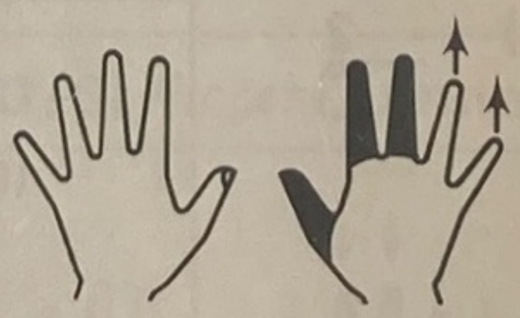
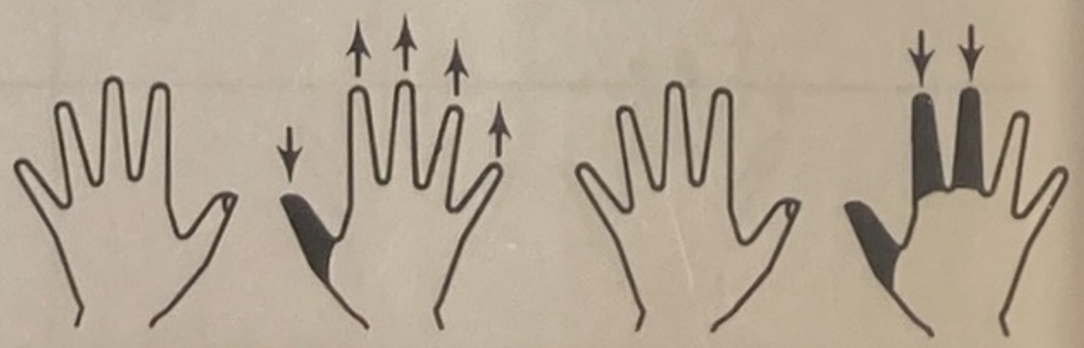
| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 3<br>(B) − 1<br>□ | (A) Press 3 one-by-one (saying) *"One, Two, Three"* | (A)  |
| | (B) Clear 1 (saying) *"One"* | (B)  |

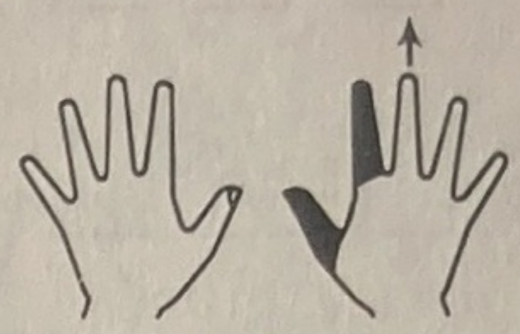
*"Read Your Fingers"* **Answer:** *"2"*

| | | |
|---|---|---|
| (A) 3<br>(B) − 2<br>□ | (A) Press 3 one-by-one (saying) *"One, Two, Three"* | (A)  |
| | (B) Clear 2 one-by-one (saying) *"One, Two"* | (B)  |

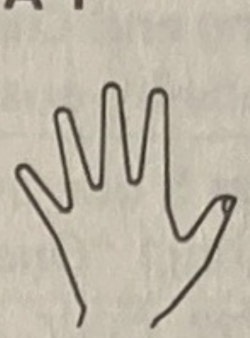
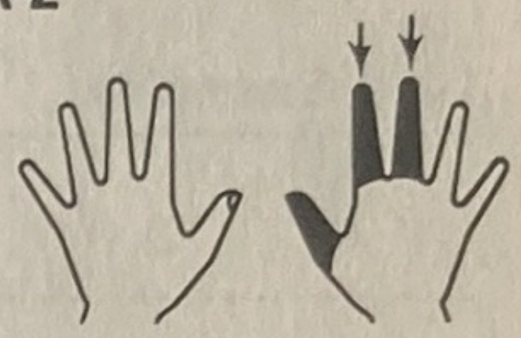
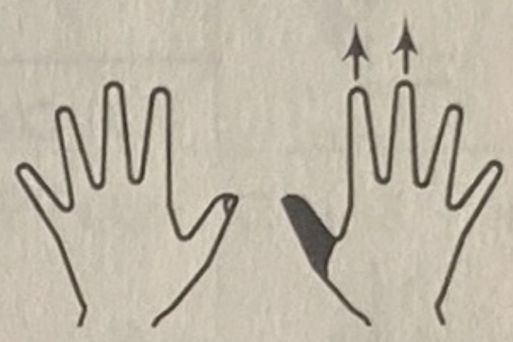
*"Read Your Fingers"* **Answer:** *"1"*

## Right Hand—Numbers 0 through 9

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 9<br>(B) − 1<br>□ | **(A) Press 9 one-by-one**<br>**A-1** (saying) *"One, Two, Three, Four, Five"*<br>**A-2** (saying) *"Six, Seven, Eight, Nine"* | **A-1** **A-2**<br>  |
| | **(B) Clear 1**<br>(saying) *"One"* | **(B)**<br>  |
| | | *"Read Your Fingers"* **Answer: *"8"*** |
| (A) 9<br>(B) − 2<br>□ | **(A) Press 9 one-by-one**<br>**A-1** (saying) *"One, Two, Three, Four, Five"*<br>**A-2** (saying) *"Six, Seven, Eight, Nine"* | **A-1** **A-2**<br>  |
| | **(B) Clear 2 one-by-one**<br>(saying) *"One, Two"* | **(B)**<br>  |
| | | *"Read Your Fingers"* **Answer: *"7"*** |
| (A) 7<br>(B) − 1<br>□ | **(A) Press 7 one-by-one**<br>**A-1** (saying) *"One, Two, Three, Four, Five"*<br>**A-2** (saying) *"Six, Seven"* | **A-1** **A-2**<br>  |

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 7<br>(B) − 1<br>□ | **(B) Clear 1**<br>(saying) *"One"* | **(B)**  |

*"Read Your Fingers"* **Answer:** *"6"*

| | | |
|---|---|---|
| (A) 7<br>(B) − 2<br>□ | **(A) Press 7 one-by-one**<br>**A-1** (saying) *"One, Two, Three, Four, Five"*<br>**A-2** (saying) *"Six, Seven"* | **A-1**  **A-2**  |
| | **(B) Clear 2 one-by-one**<br>(saying) *"One, Two"* | **(B)**  |

*"Read Your Fingers"* **Answer:** *"5"*

## One Digit Subtraction Using Only the Right Hand

| | | | | | |
|---|---|---|---|---|---|
| 3 | 3 | 3 | 3 | 4 | 1 |
| − 3 | − 2 | − 1 | − 0 | − 4 | − 1 |

## EXERCISES

Press the minuend directly. Clear the subtrahend, one-by-one, in the reverse order from the pressing order. Make sure to count as you clear. Once you have cleared the subtrahend, read your fingers for the answer.

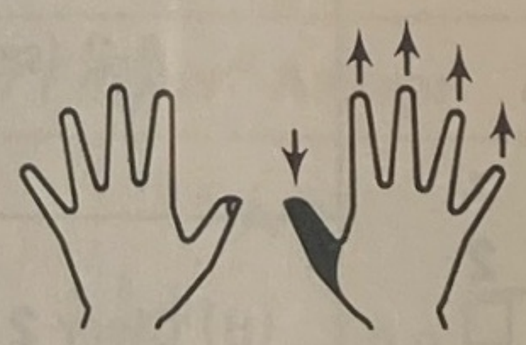
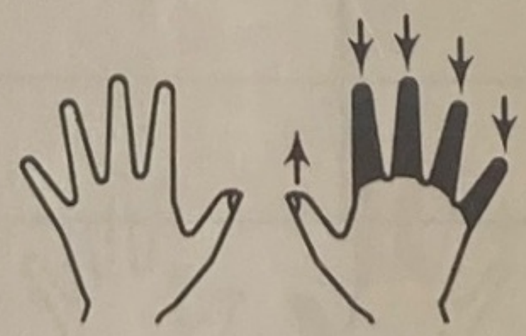
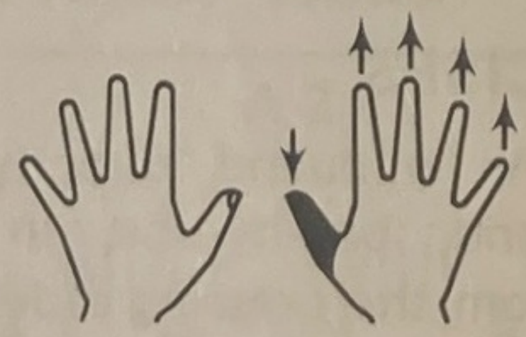
| | | | | | |
|---|---|---|---|---|---|
| 2 | 2 | 2 | 1 | 2 | 3 |
| − 2 | − 1 | − 0 | − 1 | − 1 | − 1 |

| | | | | | |
|---|---|---|---|---|---|
| 2 | 3 | 4 | 4 | 4 | 4 |
| − 2 | − 2 | − 2 | − 3 | − 1 | − 0 |

| | | | | | |
|---|---|---|---|---|---|
| 6 | 7 | 8 | 9 | 9 | 9 |
| - 1 | - 1 | - 1 | - 1 | - 3 | - 4 |
| 7 | 8 | 9 | 8 | 4 | 7 |
| - 2 | - 2 | - 2 | - 3 | - 2 | - 2 |
| 9 | 8 | 6 | 7 | 4 | 4 |
| - 2 | - 2 | - 1 | - 1 | - 3 | - 1 |

| | | |
|---|---|---|
| 2 - 1 = □ | 1 - 0 = □ |
| 3 - 0 = □ | 3 - 2 = □ |
| 1 - 1 = □ | 3 - 1 = □ |
| 3 - 3 = □ | 2 - 2 = □ |
| 8 - 2 = □ | 7 - 2 = □ |
| 7 - 1 = □ | 6 - 1 = □ |
| 8 - 3 = □ | 8 - 1 = □ |
| 9 - 1 = □ | 9 - 2 = □ |

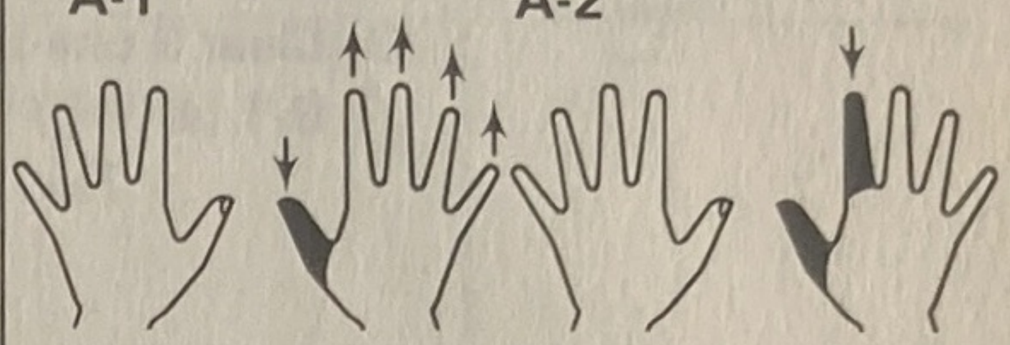
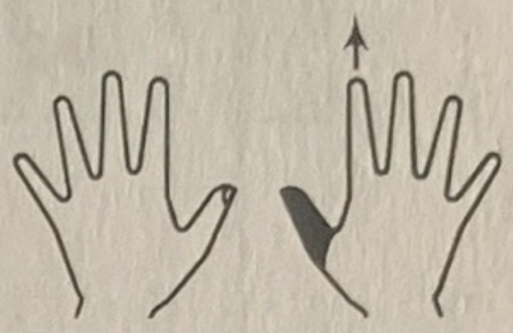
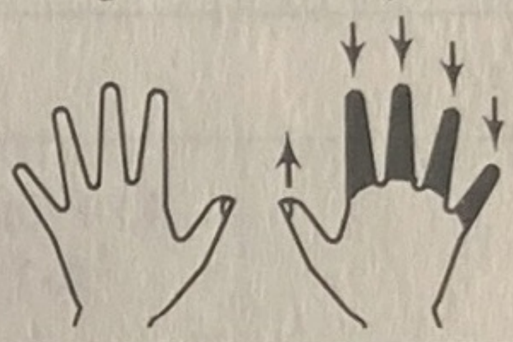
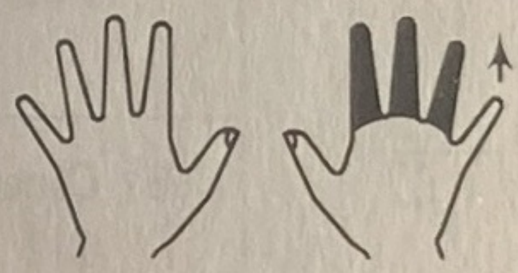
**Numbers 0 through 9 and including subtraction requiring the manipulation of** *Clear 5, Press 4*

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 5<br>(B) - 1<br>□ | **(A) Press 5 one-by-one**<br>(saying) *"One, Two, Three, Four, Five"* | **(A)**<br>  |
| | **(B) Clear 1**<br>(saying) *"One"** | **(B)** Through: (Clear 5, Press 4)<br>  |
| | | *"Read Your Fingers"* **Answer:** *"4"* |
| (A) 5<br>(B) - 2<br>□ | **(A) Press 5 one-by-one**<br>(saying) *"One, Two, Three, Four, Five"* | **(A)**<br>  |

*Inflect the voice whenever the key manipulation is made.

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 5<br>(B) - 2<br>□ | **(B) Clear 2 one-by-one**<br>**B-1** (saying) *"One"**<br>**B-2** (saying) *"Two"* | **B-1** Through: **B-2**<br>(Clear 5, Press 4)<br> |

*"Read Your Fingers"* **Answer:** *"3"*

| | | |
|---|---|---|
| | **(A) Press 6 one-by-one**<br>**A-1** (saying) *"One, Two, Three, Four, Five"*<br>**A-2** (saying) *"Six"* | **A-1** **A-2**<br>  |
| (A) 6<br>(B) - 3<br>□ | **(B) Clear 3 one-by-one**<br>**B-1** (saying) *"One"*<br>**B-2** (saying) *"Two"**<br>**B-3** (saying) *"Three"* | **B-1**<br>  <br>**B-2** Through: (Clear 5, Press 4)<br>  <br>**B-3**<br>  |

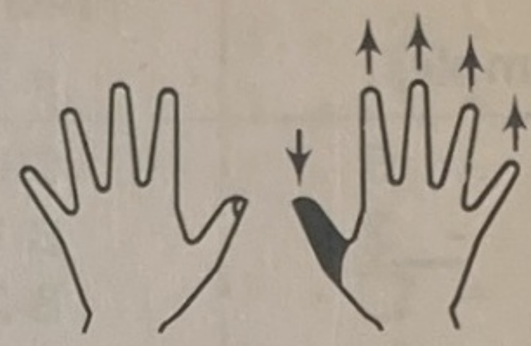
*"Read Your Fingers"* **Answer:** *"3"*

*Inflect the voice whenever this key manipulation is made.

(A) 7
(B) − 3
□

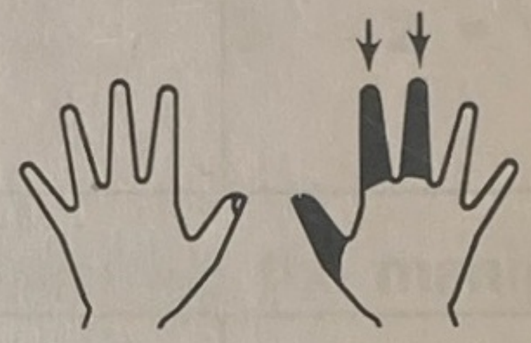
**(A) Press 7 one-by-one**
**A-1** (saying) *"One, Two, Three, Four, Five"*

**A-1**



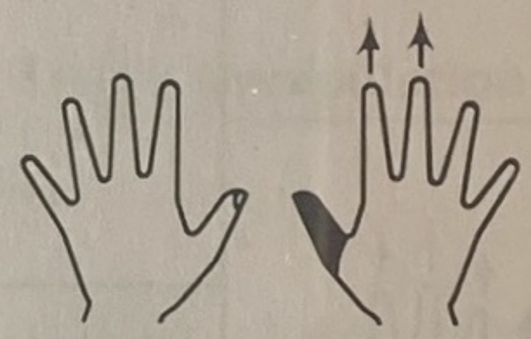
**A-2** (saying) *"Six, Seven"*

**A-2**



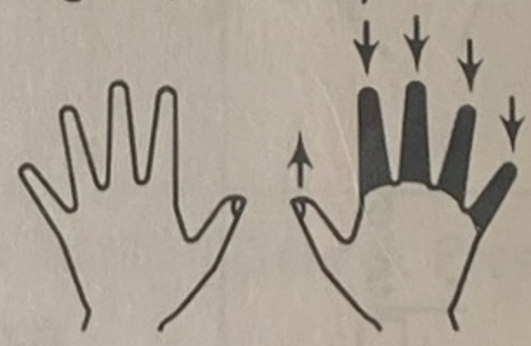
**(B) Clear 3 one-by-one**
**B-1** (saying) *"One, Two"*

**B-1**



**B-2** (saying) *"Three"**

**B-2** Through: (Clear 5, Press 4)

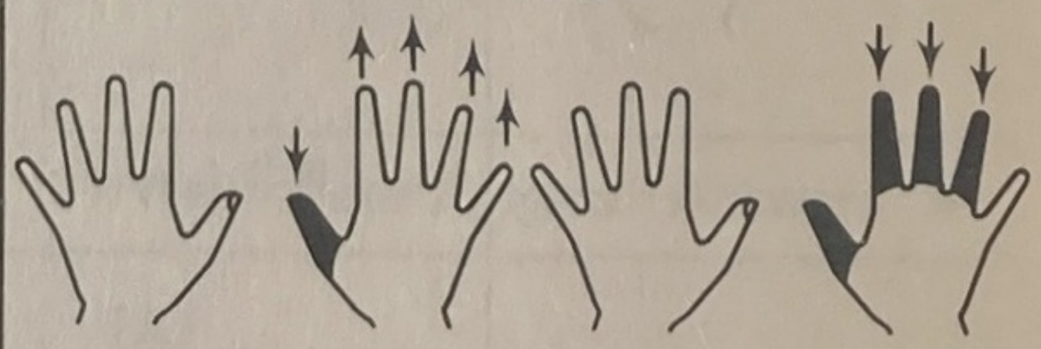


*"Read Your Fingers"* **Answer:** *"4"*

(A) 8
(B) − 5
□

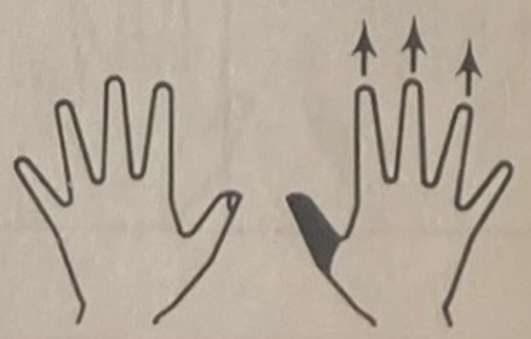
**(A) Press 8 one-by-one**
**A-1** (saying) *"One, Two, Three, Four, Five"*
**A-2** (saying) *"Six, Seven, Eight"*

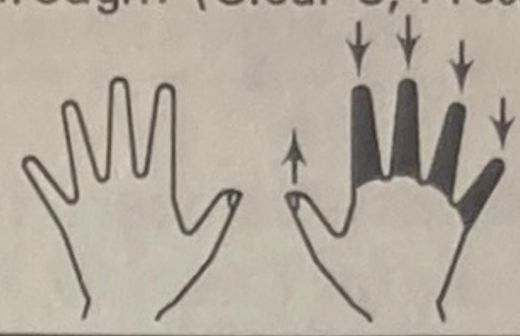
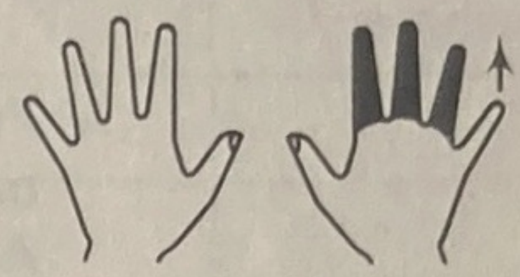
**A-1** **A-2**



**(B) Clear 5 one-by-one**
**B-1** (saying) *"One, Two, Three"*

**B-1**



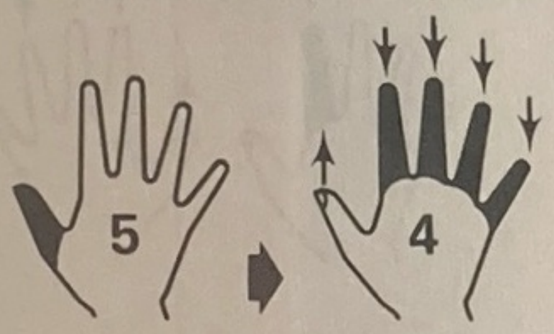
| | |
|---|---|
| **B-2** (saying) *"Four"** | **B-2** Through: (Clear 5, Press 4)  |
| **B-3** (saying) *"Five"* | **B-3**  |

*"Read Your Fingers"* **Answer:** *"3"*

*Inflect the voice whenever this key manipulation is made.

## PRACTICE EXERCISES

In these problems, for the first time, the key manipulation of Clear 5 Press 4 is called for. Inflect your voice whenever you make this manipulation. Use the HYPAI Chisanbop ™ Number Stairs for a visual display of this manipulation.




**Clear 5 & Press 4 at the Same Time**

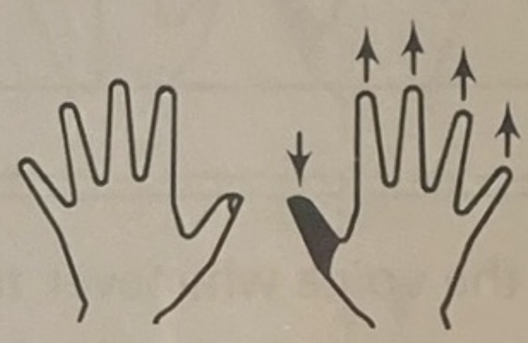
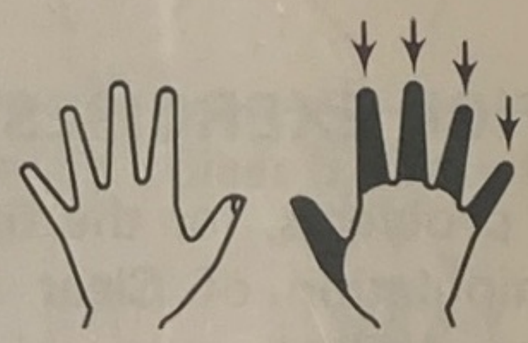
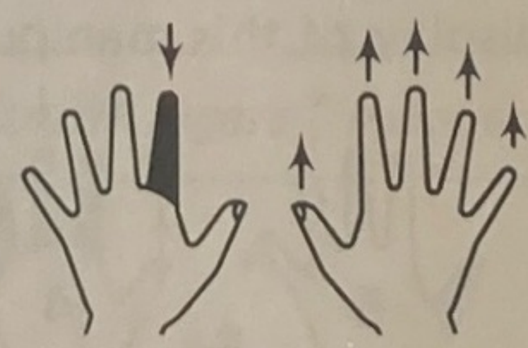
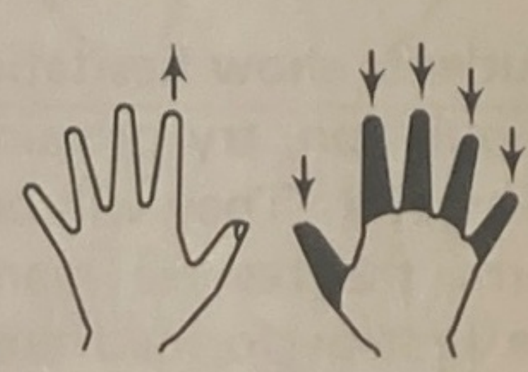
If the students show hesitation with this key manipulation, try pressing the minuend, one-by-one. They will be better able to perform the reverse manipulation if they have first performed the key manipulation of addition.

| | | | | | |
|---|---|---|---|---|---|
| 5 | 5 | 5 | 5 | 5 | 5 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 |

| | | | | | |
|---|---|---|---|---|---|
| 6 | 6 | 6 | 6 | 6 | |
| - 2 | - 3 | - 4 | - 5 | - 6 | |
| 7 | 7 | 7 | 7 | 7 | |
| - 3 | - 4 | - 5 | - 7 | - 6 | |
| 8 | 8 | 8 | 8 | 8 | |
| - 5 | - 4 | - 6 | - 7 | - 8 | |
| 9 | 9 | 9 | 9 | 9 | |
| - 5 | - 6 | - 7 | - 8 | - 9 | |
| 6 | 8 | 5 | 8 | 7 | |
| - 4 | - 7 | - 3 | - 5 | - 4 | |
| 5 | 6 | 8 | 9 | 6 | |
| - 4 | - 4 | - 4 | - 5 | - 6 | |

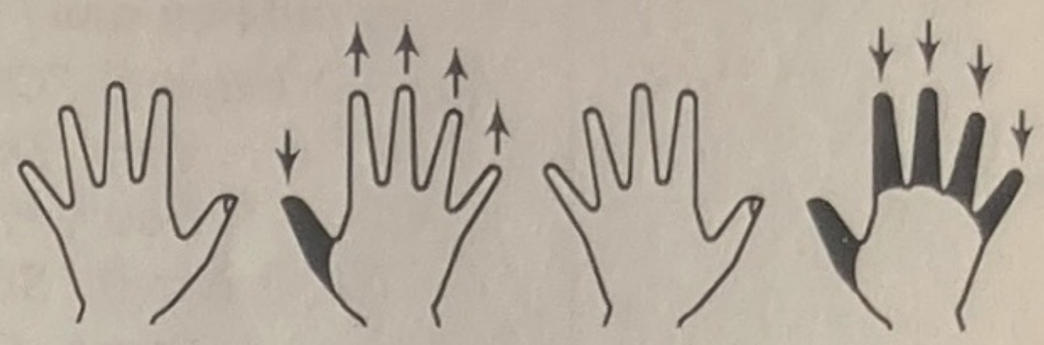
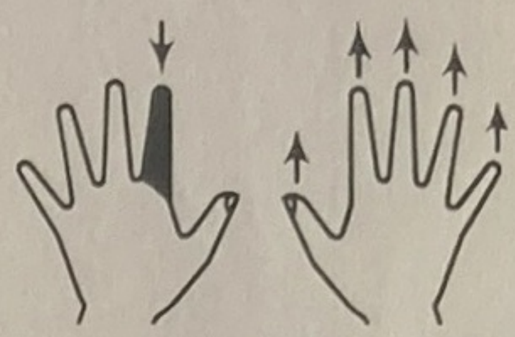
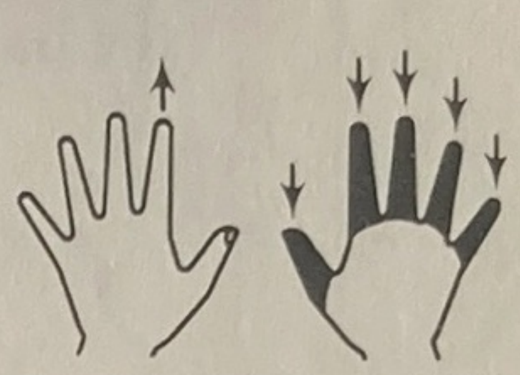
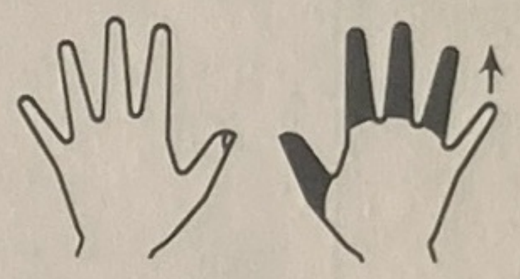
| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 6 - 3 = ☐ | | 5 - 0 = ☐ | | 5 - 5 = ☐ | | 8 - 6 = ☐ | | 8 - 8 = ☐ | |
| 5 - 3 = ☐ | | 6 - 2 = ☐ | | 9 - 5 = ☐ | | 9 - 9 = ☐ | | 9 - 8 = ☐ | |
| 5 - 4 = ☐ | | 5 - 2 = ☐ | | | | 9 - 6 = ☐ | | 8 - 7 = ☐ | |

**Both Hands—Numbers 10 through 19 requiring the manipulation of *Clear 5, Press 4* and/or *Clear 10, Press 9***

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 10<br>(B) - 1<br>☐ | **(A) Press 10 (directly) or one-by-one**<br>**A-1** (saying) *"One, Two, Three, Four, Five"* | **A-1**<br>  |
| | **A-2** (saying) *"Six, Seven, Eight, Nine"* | **A-2**<br>  |
| | **A-3** (saying) *"Onety"* (10) | **A-3**<br>  |
| | **(B) Clear 1** (saying) *"One"** | (B) Through: (Clear 10, Press 9)<br>  |

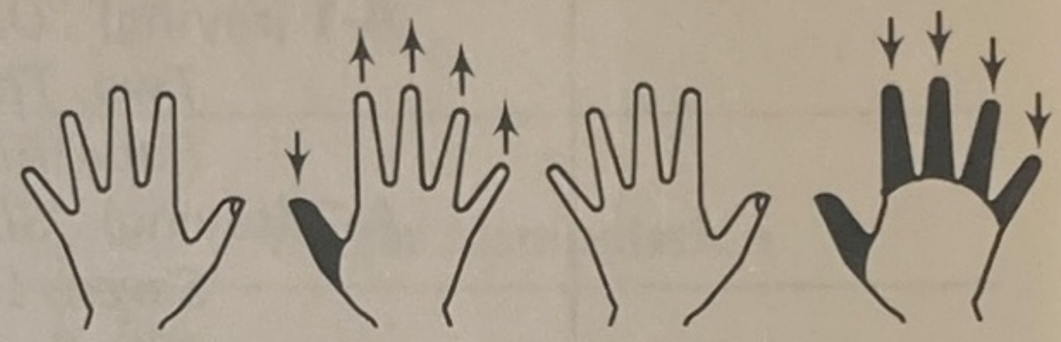
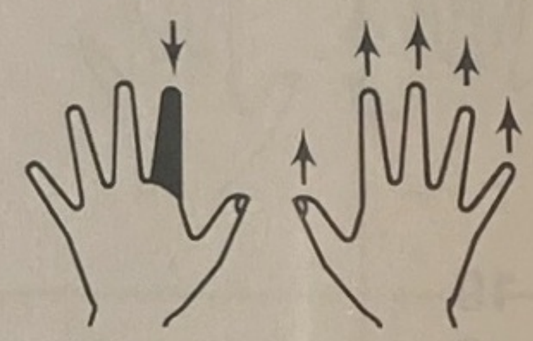
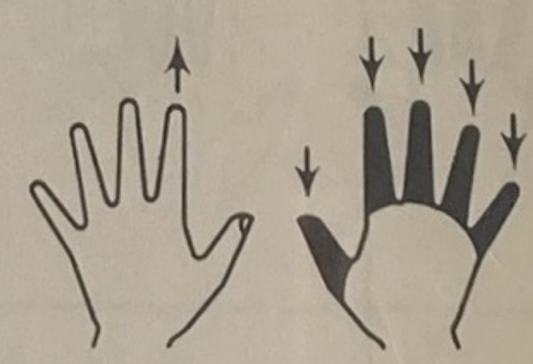
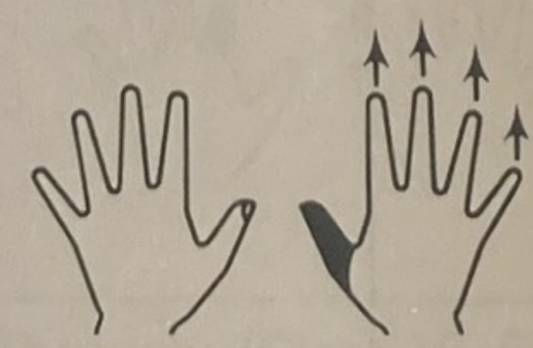
*"Read Your Fingers"* **Answer: *"9"***

*Inflect the voice at key manipulations.

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 10<br>(B) − 2<br>□ | (A) Press 10 (directly) or one-by-one<br>A-1 (saying) *"One, Two, Three, Four, Five"*<br>A-2 (saying) *"Six, Seven, Eight, Nine"* | A-1 A-2  |
| | A-3 (saying) *"Onety"* (10) | A-3  |
| | (B) Clear 2 one-by-one<br>B-1 (saying) *"One"** | B-1 Through: (Clear 10, Press 9)  |
| | B-2 (saying) *"Two"* | B-2  |

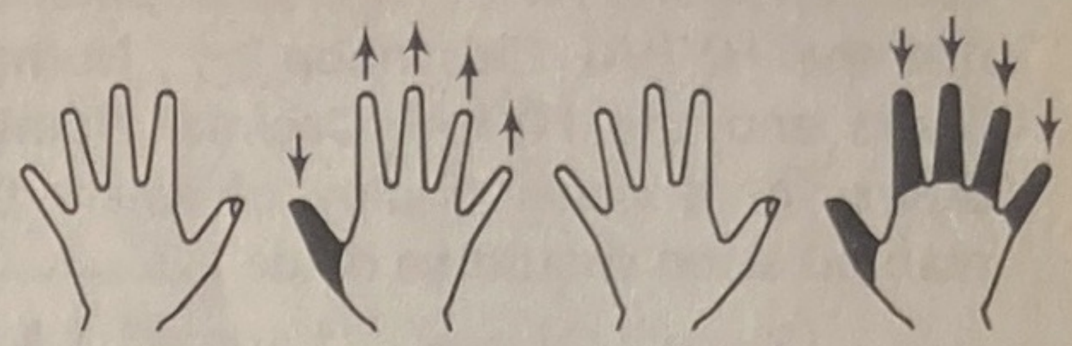
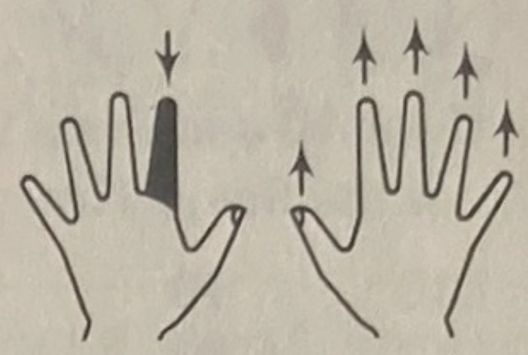
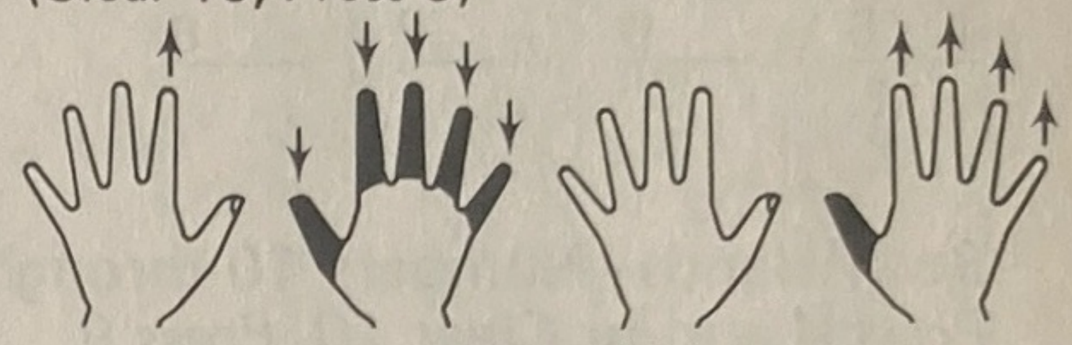
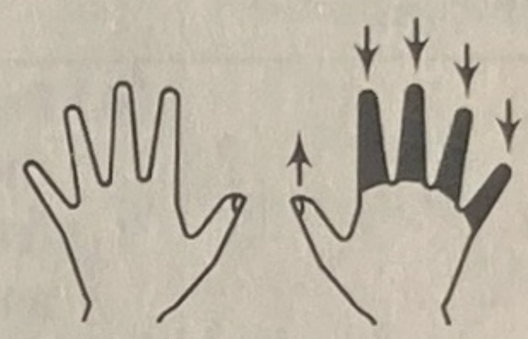
*"Read Your Fingers"* **Answer:** *"8"*

*Inflect the voice at key manipulations.

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 10<br>(B) - 5<br>□ | (A) Press 10 (directly) or one-by-one<br>A-1 (saying) *"One, Two, Three, Four, Five"*<br>A-2 (saying) *"Six, Seven, Eight, Nine"* | A-1 A-2<br>  |
| | A-3 (saying) *"Onety"* (10) | A-3<br>  |
| | (B) Clear 5 one-by-one<br>B-1 (saying) *"One"** | B-1 Through: (Clear 10, Press 9)<br>  |
| | B-2 (saying) *"Two, Three, Four, Five"* | B-2<br>  |

*"Read Your Fingers"* **Answer: *"5"***

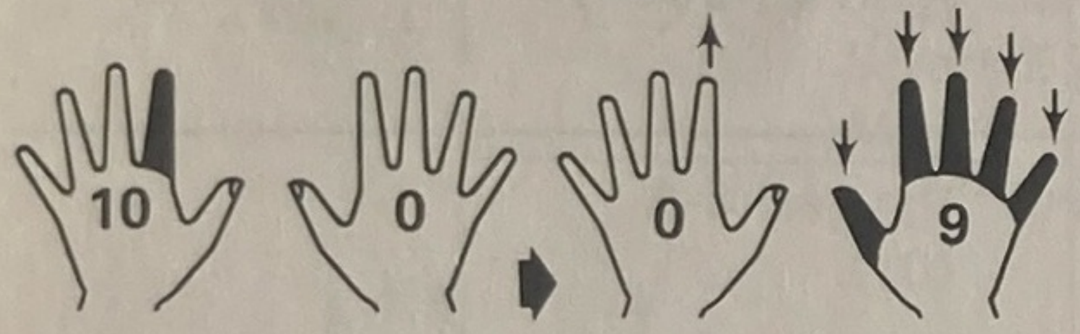
*Inflect the voice at key manipulations.

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 10<br>(B) − 6<br>□ | (A) Press 10 (directly) or one-by-one<br>**A-1** (saying) *"One, Two, Three, Four, Five"*<br>**A-2** (saying) *"Six, Seven, Eight, Nine"* | **A-1** **A-2**<br>  |
| | **A-3** (saying) *"Onety"* (10) | **A-3**<br>  |
| | **(B) Clear 6 one-by-one**<br>**B-1** (saying) *"One"**<br>**B-2** (saying) *"Two, Three, Four, Five"* | **B-1** Through: (Clear 10, Press 9) **B-2**<br>  |
| | **B-3** (saying) *"Six"** | **B-3** Through: (Clear 5, Press 4)<br>  |
| | | *"Read Your Fingers"* **Answer:** *"4"* |

*Inflect the voice at key manipulations.

## PRACTICE EXERCISES

The second key manipulation of Clear 10 Press 9 is introduced here. If the students have difficulty with this manipulation, use the HYPAI Chisanbop ™ Number Stairs and the HYPAI Decimal Number Stairs for a visual display of where this manipulation should be made.



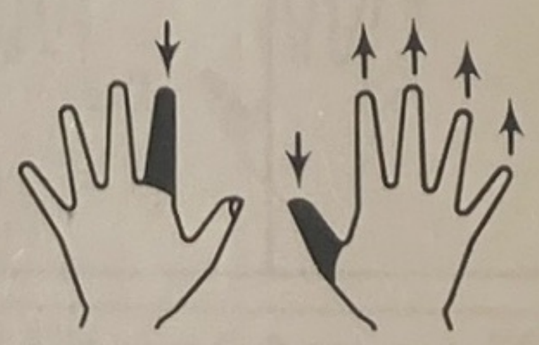
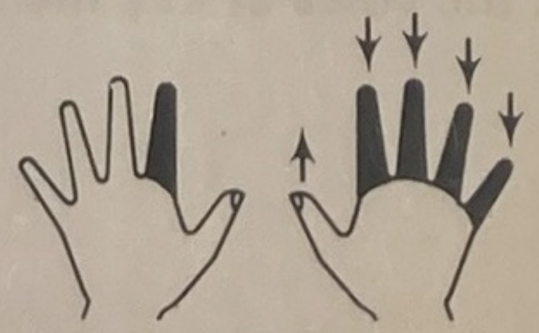

**Clear 10 and Press 9 at the Same Time**

| | | | |
|---|---|---|---|
| 10 | 10 | 10 | 10 |
| - 1 | - 2 | - 3 | - 4 |
| 10 | 10 | 10 | 10 |
| - 5 | - 6 | - 7 | - 8 |

| | | | |
|---|---|---|---|
| 10 | 10 | 10 | 10 |
| - 9 | - 4 | - 1 | - 5 |
| 10 | 10 | 10 | 10 |
| - 8 | - 9 | - 3 | - 4 |
| 10 | 10 | 10 | 10 |
| - 5 | - 6 | - 7 | - 8 |

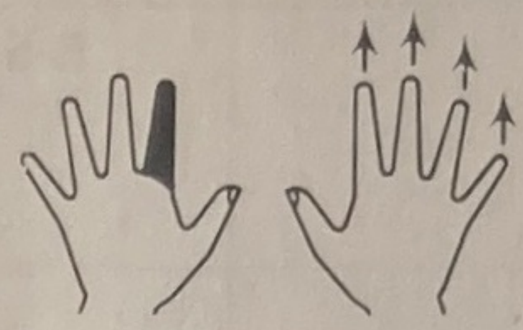
| | |
|---|---|
| 10 - 3 = ☐ | 10 - 4 = ☐ |
| 10 - 8 = ☐ | 10 - 5 = ☐ |
| 10 - 2 = ☐ | 10 - 6 = ☐ |
| 10 - 9 = ☐ | 10 - 7 = ☐ |
| 10 - 9 = ☐ | 10 - 2 = ☐ |
| 10 - 7 = ☐ | 10 - 1 = ☐ |
| 10 - 4 = ☐ | 10 - 5 = ☐ |
| | 10 - 6 = ☐ |

**Both Hands—Numbers 10 through 19, requiring the manipulation of *Clear 5, Press 4* and/or *Clear 10, Press 9***

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 15<br>(B) - 6<br>☐ | **(A) Press 10 directly and then 5 more one-by-one** (saying) *"Onety"* (10) and then *"One, Two, Three, Four, Five"* | **(A)**  |
| | **(B) Clear 6 one-by-one**<br>**B-1** (saying) *"One"** | **B-1** Through: (Clear 5, Press 4)  |

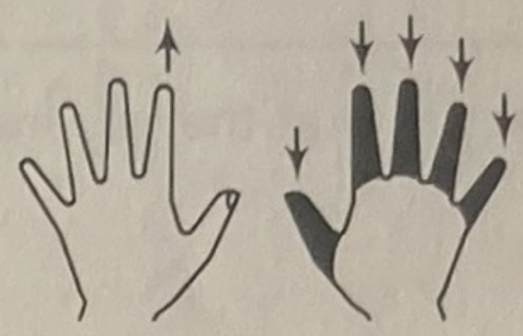
**B-2** (saying) *"Two, Three, Four, Five"*

**B-2**



**B-3** (saying) *"Six"**

**B-3** Through: (Clear 10, Press 9)



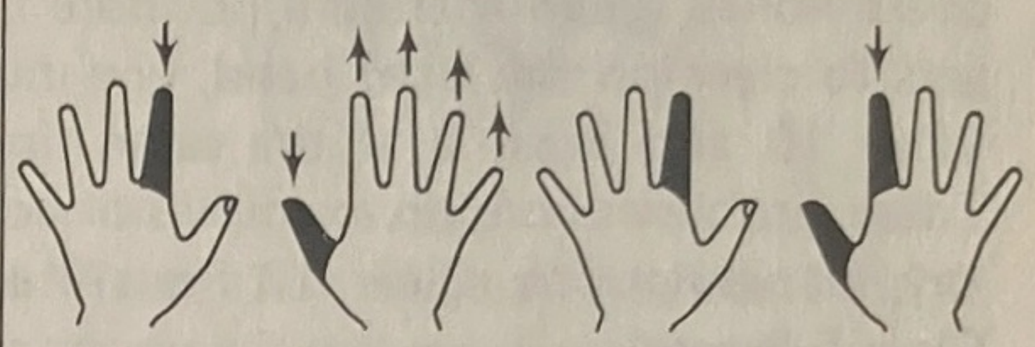
*"Read Your Fingers"* **Answer:** *"9"*

**(A) Press 10 directly and then 6 more one-by-one**

**A-1** (saying) *"Onety"* (10) and then *"One, Two, Three, Four, Five"*

**A-2** (saying) *"Six"*

**A-1** **A-2**



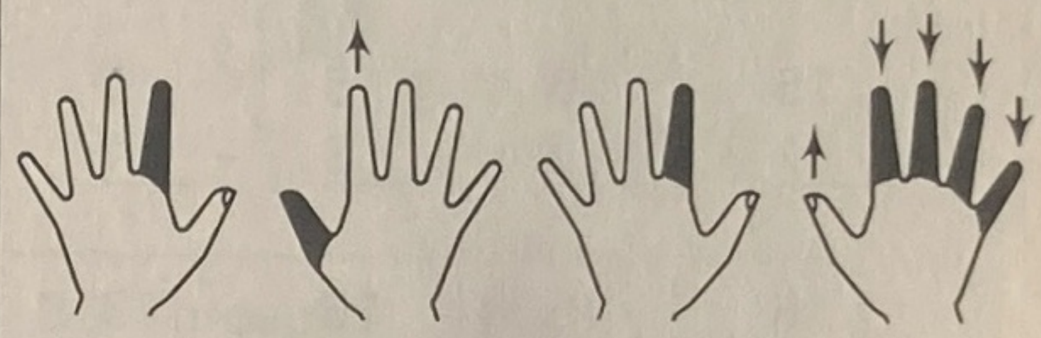
**(A)** 16
**(B)** − 9
□

**(B) Clear 9 one-by-one**

**B-1** (saying) *"One"*

**B-2** (saying) *"Two"**

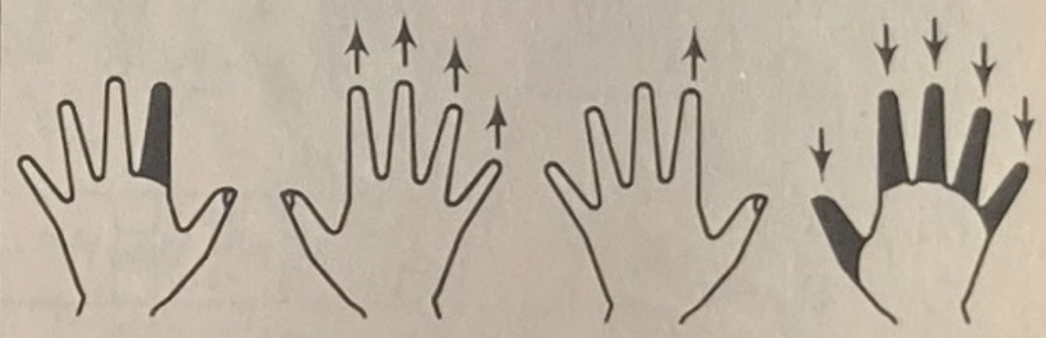
**B-1** **B-2** Through: (Clear 5, Press 4)



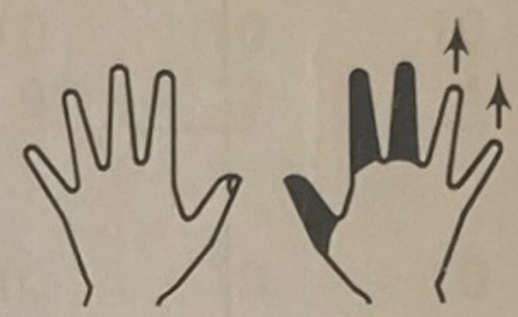
**B-3** (saying) *"Three, Four, Five, Six"*

**B-4** (saying) *"Seven"**

**B-3** **B-4** Through: (Clear 10, Press 9)



*Inflect the voice at the key manipulations.

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| | **B-5** (saying) *"Eight, Nine"* | **B-5**  |

*"Read Your Fingers"* **Answer:** *"7"*

*Inflect the voice at the key manipulations.

## EXERCISES

Press the entire minuend and then clear, one-by-one. When there are no more fingers to clear on the right hand, you must Clear 10 and Press 9 at the same time. These problems contain examples of both key manipulations, Clear 10 Press 9 and Clear 5 Press 4.

| | | | |
|---|---|---|---|
| 15 | 15 | 15 | 15 |
| - 6 | - 7 | - 8 | - 9 |

| | | | |
|---|---|---|---|
| 15 | 15 | 15 | 15 |
| - 1 | - 2 | - 3 | - 4 |

| | | | |
|---|---|---|---|
| 15 | 15 | 15 | 15 |
| - 3 | - 8 | - 7 | - 5 |

| | | | |
|---|---|---|---|
| 15 | 15 | 15 | 15 |
| - 9 | - 2 | - 4 | - 6 |

15 - 4 = ☐
15 - 9 = ☐
15 - 7 = ☐
15 - 8 = ☐

15 - 2 = ☐
15 - 5 = ☐
15 - 6 = ☐
15 - 3 = ☐

15 - 2 = ☐
15 - 8 = ☐
15 - 6 = ☐
15 - 4 = ☐

15 - 3 = ☐
15 - 7 = ☐
15 - 9 = ☐
15 - 5 = ☐

Do not perform columnar subtraction or regrouping. That will come later. Thus, you should press the entire minuend and proceed to clear, one-by-one. Whenever a key manipulation is called for, inflect your voice as you perform the manipulation.

| | | | |
|---|---|---|---|
| 16 | 16 | 16 | 17 |
| - 7 | - 8 | - 9 | - 8 |

| | | | |
|---|---|---|---|
| 17 | 18 | 15 | 15 |
| - 9 | - 9 | - 6 | - 7 |

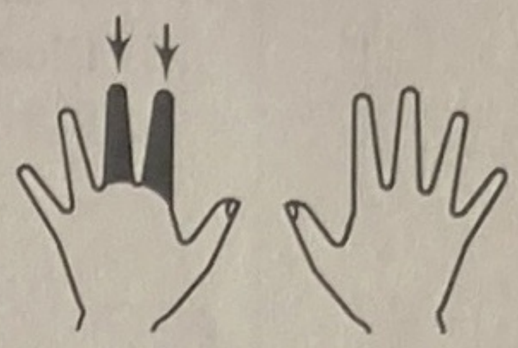
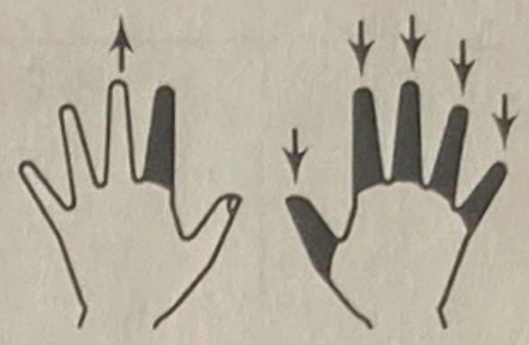
| | | | |
|---|---|---|---|
| 16 | 17 | 18 | 19 |
| - 7 | - 8 | - 9 | - 7 |

| | | | |
|---|---|---|---|
| 16 | 17 | 18 | 19 |
| - 9 | - 9 | - 9 | - 9 |

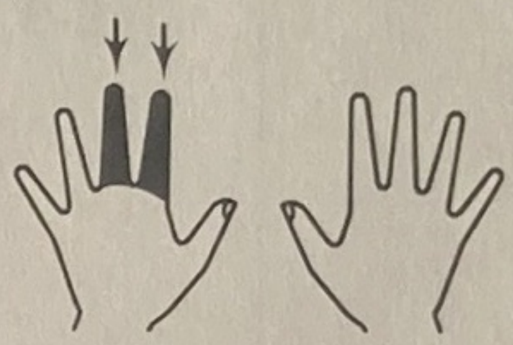
| | | |
|---|---|---|
| 15 - 8 = ☐ | 15 - 7 = ☐ | 16 - 8 = ☐ |
| 15 - 6 = ☐ | 16 - 7 = ☐ | 17 - 8 = ☐ |
| 15 - 9 = ☐ | 16 - 9 = ☐ | 17 - 9 = ☐ |
| | | 18 - 9 = ☐ |

| | | |
|---|---|---|
| 19 - 9 = ☐ | 17 - 8 = ☐ | |
| 15 - 8 = ☐ | 18 - 8 = ☐ | |
| 16 - 8 = ☐ | 19 - 8 = ☐ | |

## Both Hands—Numbers 20 through 29

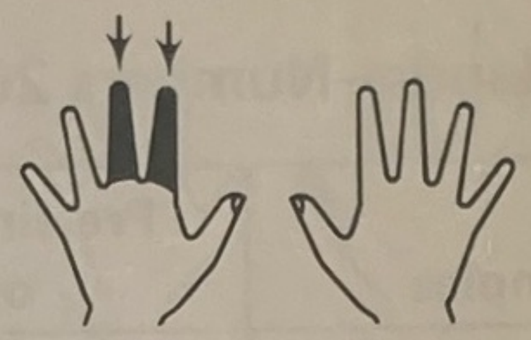
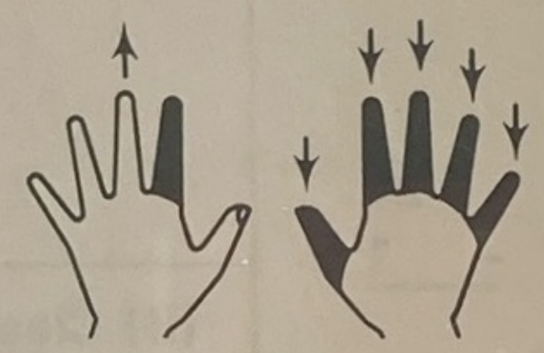
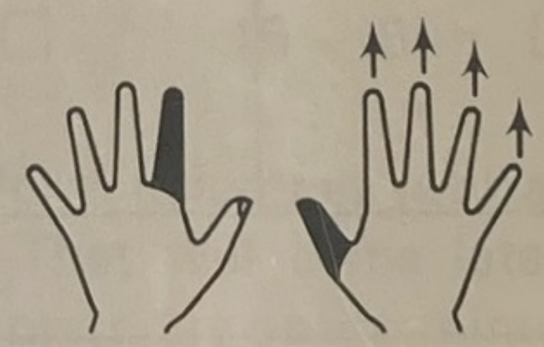
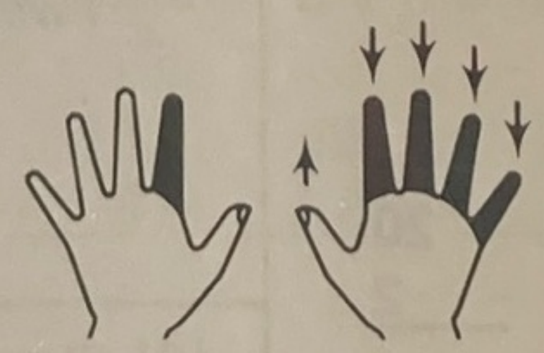
| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 20<br>(B) - 1 | **(A) Press 20 ten-by-ten**<br>(saying) *"Onety, Twenty"* | **(A)**<br>  |
| | **(B) Clear 1**<br>(saying) *"One"** | **(B)** Through: (Clear 10, Press 9)<br>  |

*"Read Your Fingers"* **Answer:** *"19"*

| | | |
|---|---|---|
| (A) 20<br>(B) - 2 | **(A) Press 20 ten-by-ten**<br>(saying) *"Onety, Twenty"* | **(A)**<br>  |
| | **(B) Clear 2 one-by-one**<br>**B-1** (saying) *"One"**<br>**B-2** (saying) *"Two"* | **B-1** Through: **B-2**<br>(Clear 10, Press 9)<br> |

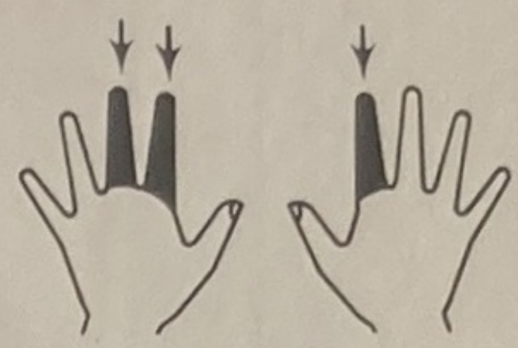
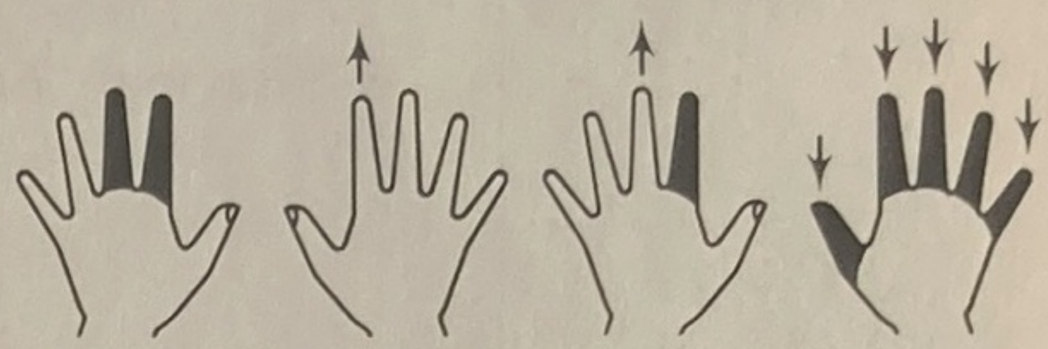
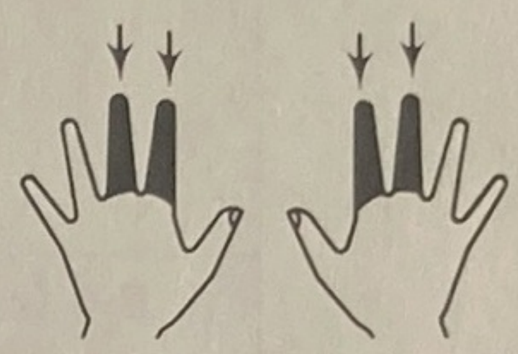
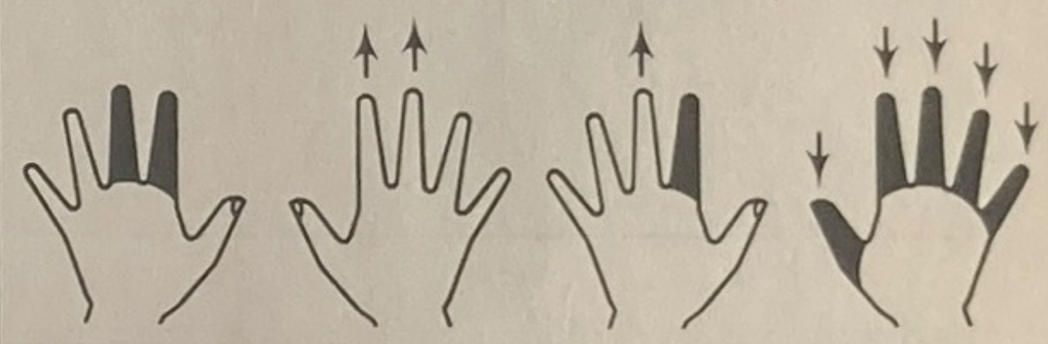
*"Read Your Fingers"* **Answer:** *"18"*

*Inflect the voice at key manipulations.

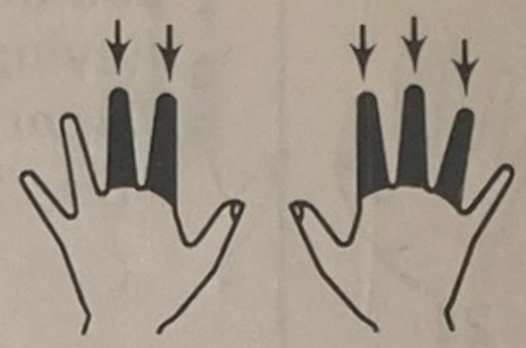
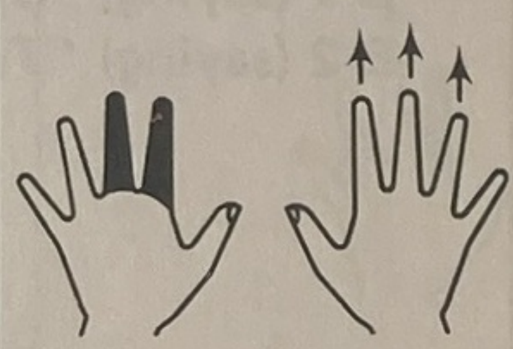
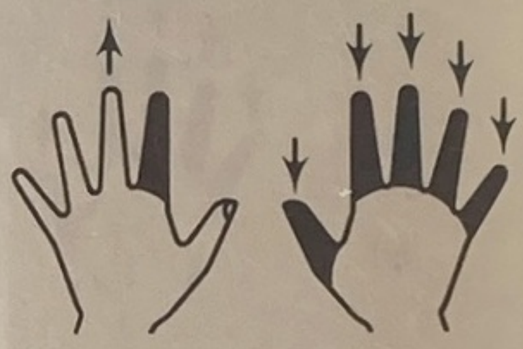
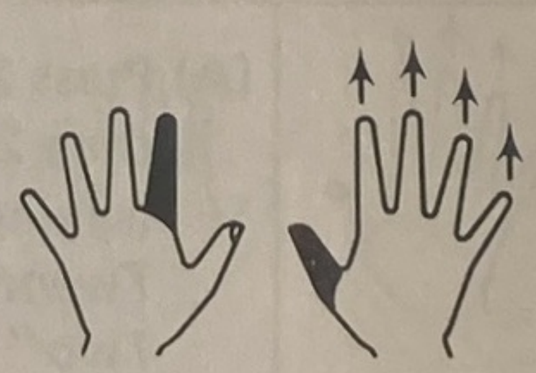
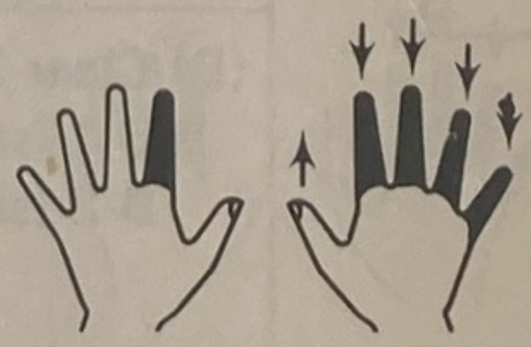
| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 20<br>(B) − 6 | (A) Press 20 ten-by-ten (saying) *"Onety, Twenty"* | (A)  |
| | (B) Clear 6 one-by-one<br>B-1 (saying) *"One"** | B-1 Through: (Clear 10, Press 9)  |
| | B-2 (saying) *"Two, Three, Four, Five"* | B-2  |
| | B-3 (saying) *"Six"** | B-3 Through: (Clear 5, Press 4)  |

*"Read Your Fingers"* **Answer:** *"14"*

*Inflect the voice at key manipulations.

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 21<br>(B) - 2 | **(A) Press 20 ten-by-ten and then 1 more** (saying) *"Onety, Twenty"* and then *"One"* | (A)  |
| | **(B) Clear 2 one-by-one**<br>**B-1** (saying) *"One"*<br>**B-2** (saying) *"Two"** | **B-1** **B-2** Through: (Clear 10, Press 9)  |
| | | *"Read Your Fingers"* **Answer:** *"19"* |
| (A) 22<br>(B) - 3 | **(A) Press 20 ten-by-ten and then 2 more one-by-one** (saying) *"Onety, Twenty"* and then *"One, Two"* | (A)  |
| | **(B) Clear 3 one-by-one**<br>**B-1** (saying) *"One, Two"*<br>**B-2** (saying) *"Three"** | **B-1** **B-2** Through: (Clear 10, Press 9)  |
| | | *"Read Your Fingers"* **Answer:** *"19"* |

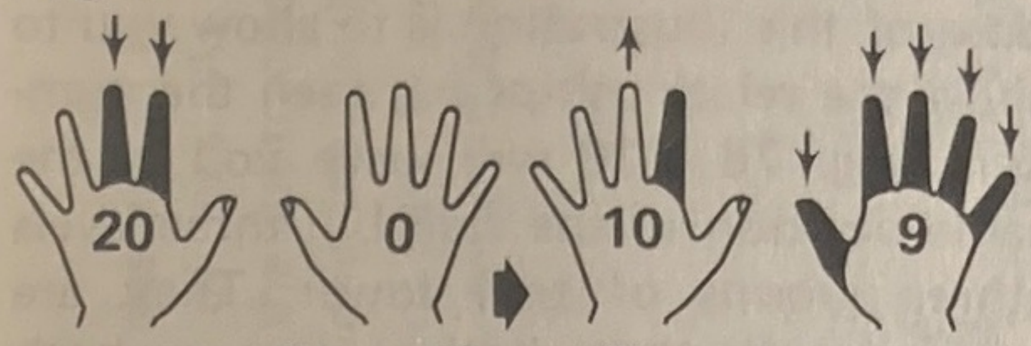
*Inflect the voice at key manipulations.

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 23<br>(B) - 9 | (A) Press 20 ten-by-ten and then 3 more one-by-one (saying) *"Onety, Twenty"* and then *"One, Two, Three"* | (A)  |
| | (B) Clear 9 one-by-one<br>B-1 (saying) *"One, Two, Three"*<br>B-2 (saying) *"Four"** | B-1  <br>B-2 Through: (Clear 10, Press 9)  |
| | B-3 (saying) *"Five, Six, Seven, Eight"* | B-3  |
| | B-4 (saying) *"Nine"** | B-4 Through: (Clear 5, Press 4)  |
| | | *"Read Your Fingers"* **Answer:** *"14"* |

*Inflect the voice at key manipulations.

## PRACTICE EXERCISES

Press the minuend directly, or ten-by-ten. When performing a Clear 10 Press 9 manipulation, be careful of which finger is cleared. The 20th finger (left middle finger) should be cleared, not the left index finger.




**Clear 10 and Press 9 at the Same Time**

The left hand is cleared in the reverse order from addition. Thus, if 30 is pressed, the 30th finger would be cleared ahead of the 20th finger and the 20th finger would be cleared ahead of the 10th finger.

| | | | |
|---|---|---|---|
| 20 | 20 | 20 | 20 |
| - 1 | - 2 | - 3 | - 4 |
| 20 | 20 | 20 | 20 |
| - 5 | - 6 | - 7 | - 8 |
| 20 | 20 | 20 | 20 |
| - 4 | - 9 | - 3 | - 5 |
| 20 | 20 | 20 | 20 |
| - 7 | - 8 | - 9 | - 3 |
| 20 | 20 | 20 | 20 |
| - 4 | - 5 | - 6 | - 7 |

| | | | |
|---|---|---|---|
| 21 | 22 | 23 | 24 |
| - 4 | - 4 | - 4 | - 4 |
| 25 | 26 | 27 | 28 |
| - 8 | - 8 | - 8 | - 8 |
| 28 | 27 | 26 | 25 |
| - 9 | - 9 | - 9 | - 9 |
| 21 | 27 | 25 | 24 |
| - 8 | - 9 | - 6 | - 5 |

| | | | |
|---|---|---|---|
| 23 | 28 | 22 | 26 |
| - 4 | - 3 | - 1 | - 2 |

| | |
|---|---|
| 20 - 4 = ☐ | 27 - 8 = ☐ |
| 20 - 8 = ☐ | 25 - 8 = ☐ |
| 20 - 6 = ☐ | 23 - 4 = ☐ |
| 20 - 3 = ☐ | 21 - 4 = ☐ |
| 20 - 5 = ☐ | 28 - 8 = ☐ |
| 20 - 4 = ☐ | 21 - 9 = ☐ |
| 20 - 5 = ☐ | 22 - 4 = ☐ |
| 20 - 1 = ☐ | 23 - 6 = ☐ |
| 20 - 6 = ☐ | 24 - 8 = ☐ |
| 20 - 2 = ☐ | 25 - 1 = ☐ |
| 20 - 8 = ☐ | 26 - 2 = ☐ |
| 20 - 3 = ☐ | 27 - 8 = ☐ |
| 20 - 7 = ☐ | 28 - 7 = ☐ |

## Decimal Structure in Subtraction

Once again the Chisanbop ™ manipulations provide valuable insights into the decimal system. When called to make the manipulation of Clear 10 Press 9 e.g. when subtracting 1 from 10, you are, in effect moving from a group of tens into the units' column. Eliminating a 10 from the left hand and pressing all of the units on the right hand is a fundamental movement which can be used to teach the fact that you are moving from a 2 digit number to a 1 digit number or reducing the tens by one ten.

The following illustrations contain another set of HYPAI Number Stairs. The difference between these stairs and those of addition is the fact that the directions are opposite. Just as the finger manipulations in subtraction are the exact opposite of those in addition, so too the visual representations ought to be in opposite directions. Eventually these visual and physical impressions will bring about the intellectual discovery that subtraction is indeed the inverse operation of addition.

The first illustration shows the Decimal Number Stairs as an adaptation of the number line. You can teach number theory and subtraction on them the same way that you would on the number line. Thus in the problem 13 - 5, you would count back 5 units from 13 verbally saying one, two, three, four, five and reach the number 8. The adaptation allows you to recognize when you are moving into the set of units only.

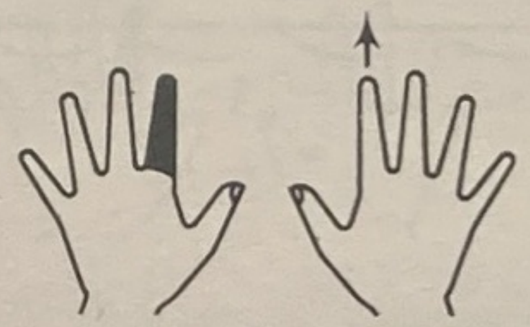
The second illustration presents the Decimal Number Stairs in sets of tens. The purpose of this illustration is to allow you to show the relationships between the numbers, e.g. 78 - 30 will leave you in the same position of the stair but three levels (three groups of ten) lower. They are presented so that children can see both the horizontal and vertical relationships in the decimal system. Finally they emphasize the key manipulation of Clear 10 Press 9 and how that relates to the concept of place value.

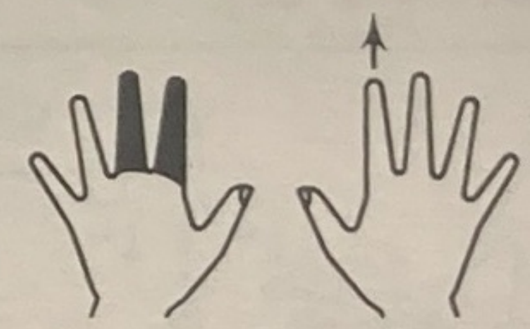
The third and fourth illustrations present the HYPAI Chisanbop ™ Number Stairs. These stairs are basically the same as the Decimal Number Stairs but they have added descents wherever the key manipulation of Clear 5 Press 4 are called for. As mentioned in addition the number 5 is a rather important number. All of the major currencies of the world have denominations of 5 in both their coins and their paper money. It is a natural number to begin counting from in both addition and subtraction. The Chisanbop ™ Number Stairs enable you to find that place visually.

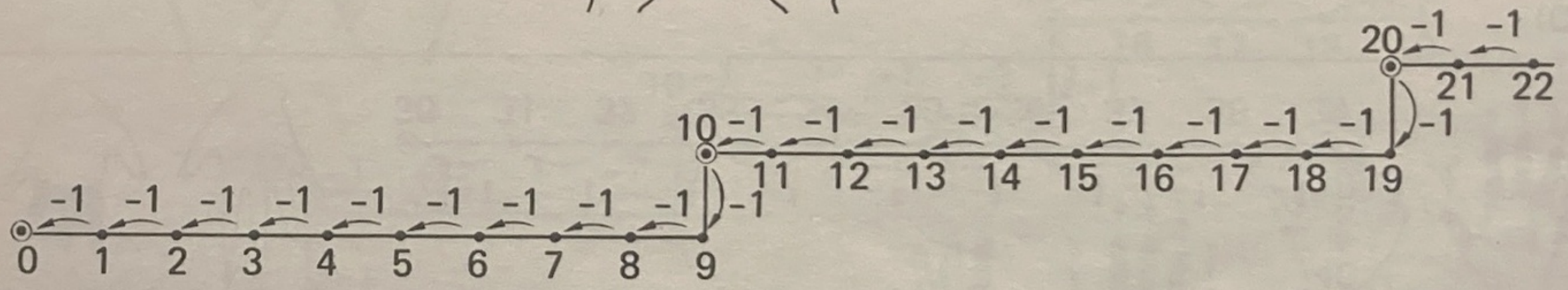
## HYPAI DECIMAL NUMBER STAIRS

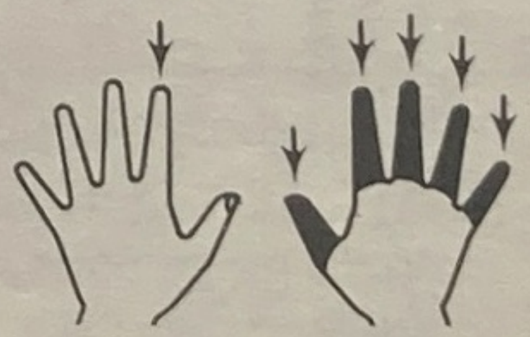
1. These stairs are read from right to left. When there is room in the classroom for a more extensive display, one continuous line from 100 to 0 should be shown.

2. Practice problems such as 15 – 7, 22 – 8, 35 – 9, etc. Point out what happens decimally whenever you come to a descent in the stairs, i.e. you are losing a group of tens. As you practice these problems show the subtraction visually, by moving the hand along the stairs. While using these stairs, you do not have to use clearing manipulations. You are concentrating here on a visual approach.

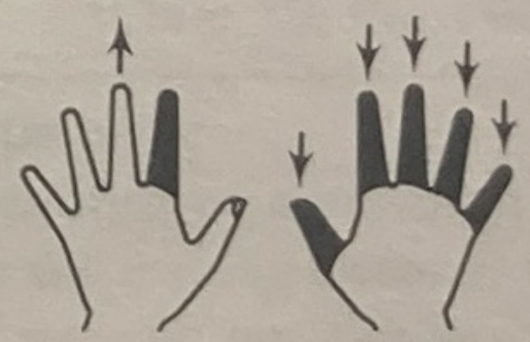






**Clear 10, Press 9**

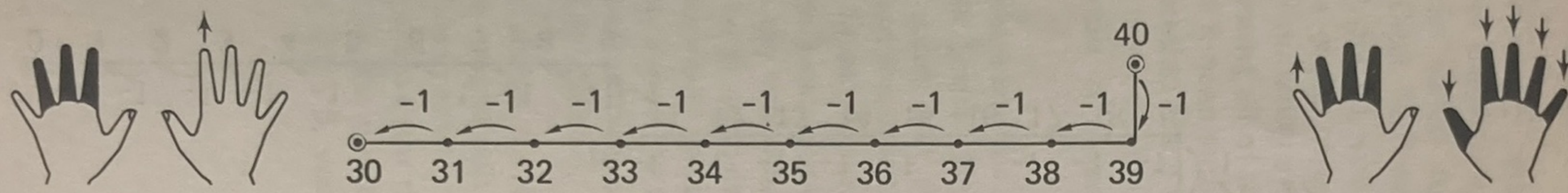


**Clear 10, Press 9**

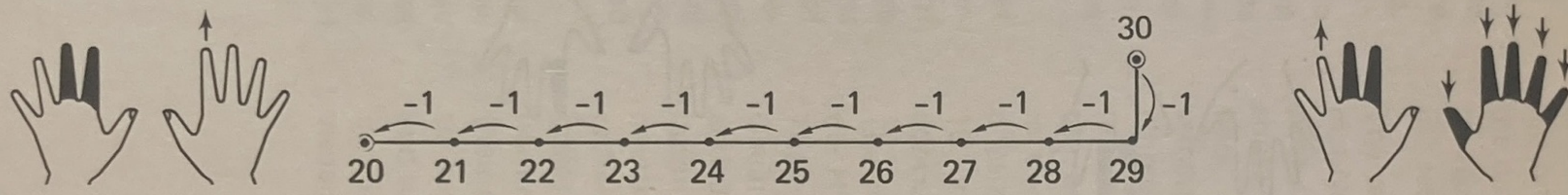
## HYPAI DECIMAL NUMBER STAIRS

These stairs are constructed to show the relationships between numbers. Practice problems such as 30 – 20, 30 – 10, and problems such as 35 – 30 or 27 – 10. To achieve this visual display, we have had to discontinue the line at 10, 20, and 30. Students should know that wherever they see a small circle at the end of the stair that the stair should be connected to the circle at the other side of the page.

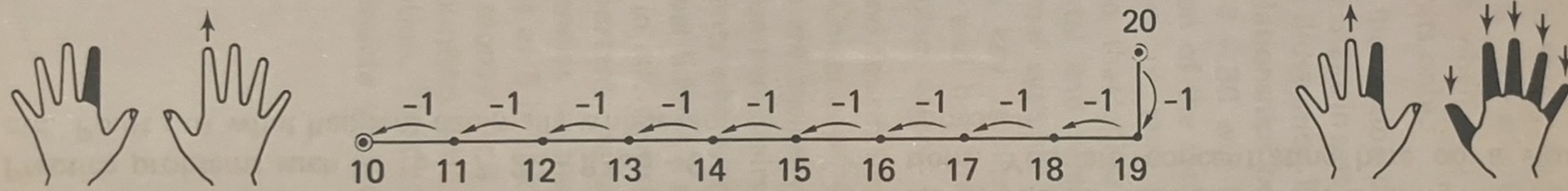
**(Clear 10, Press 9)**



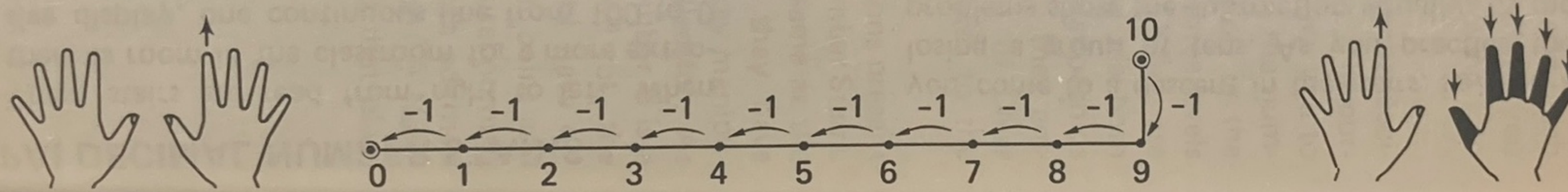

**(Clear 10, Press 9)**




**(Clear 10, Press 9)**

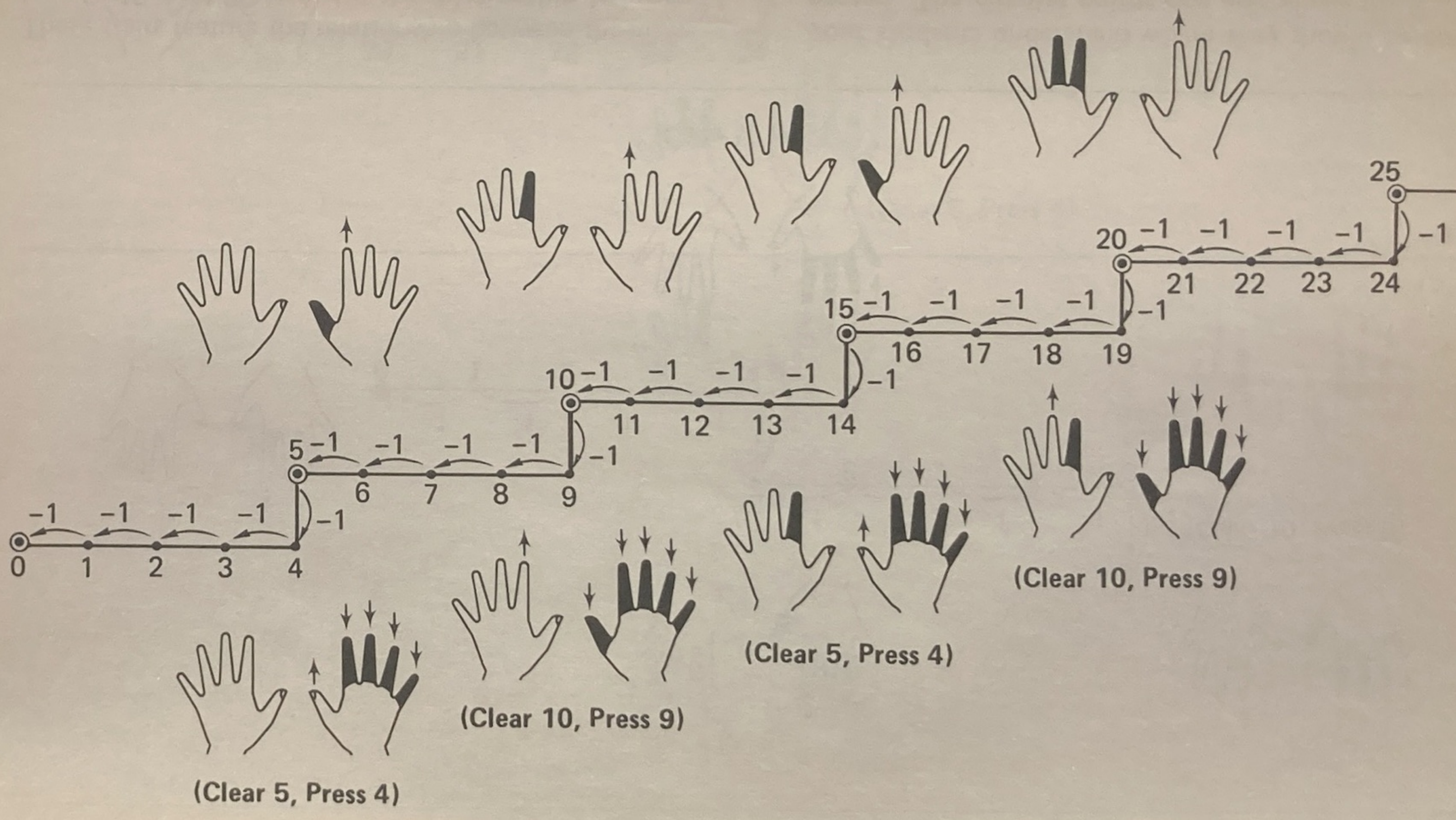


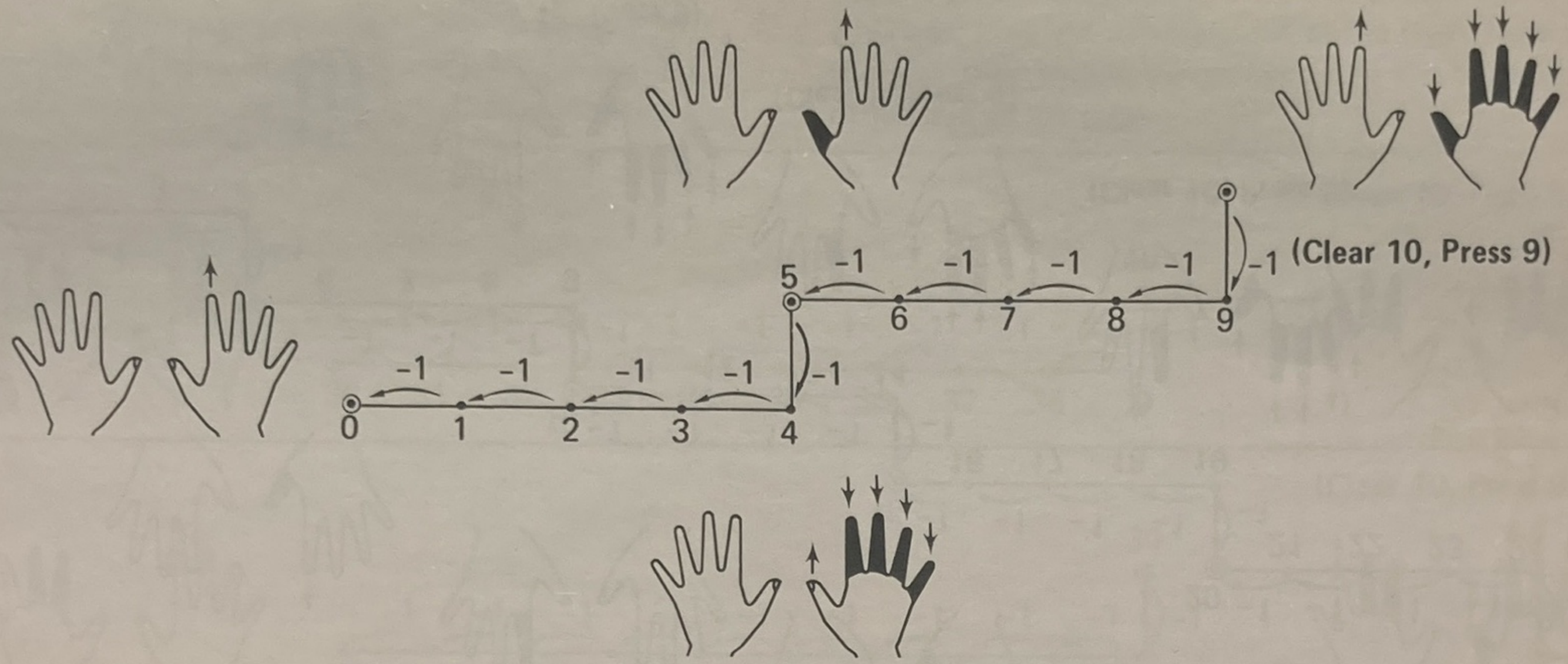

**(Clear 10, Press 9)**



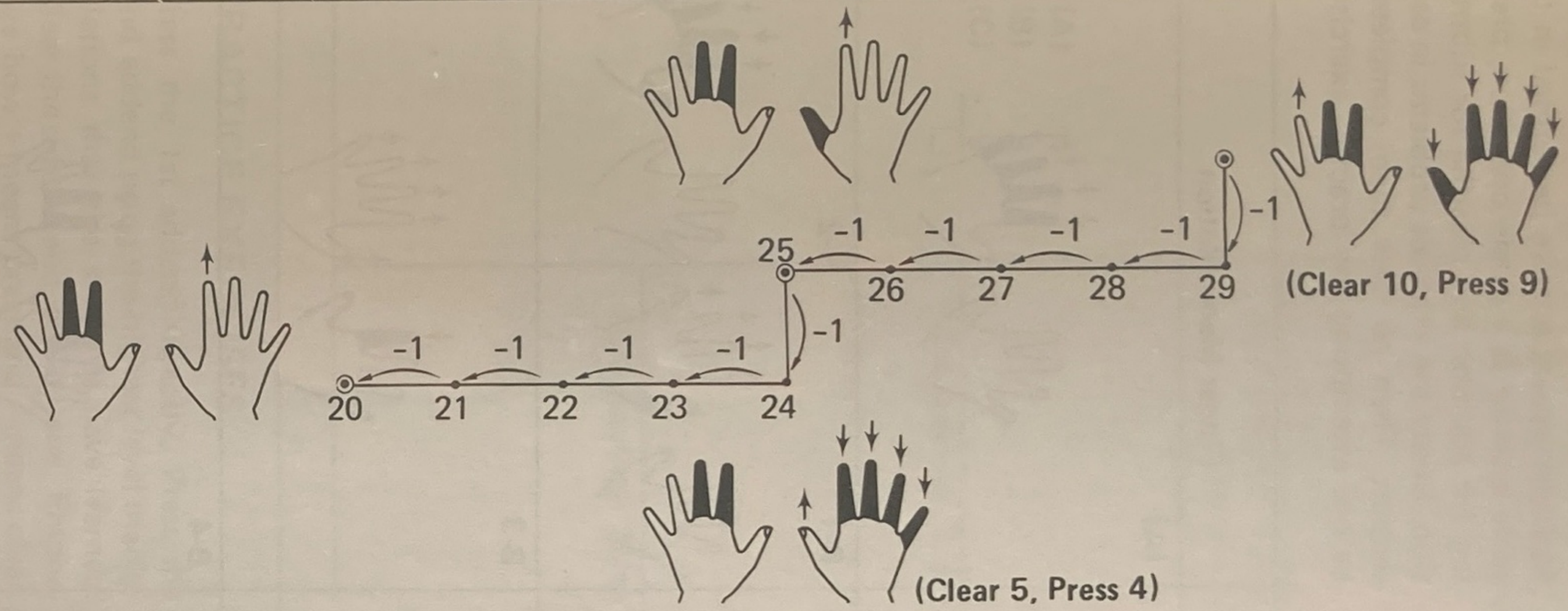
## HYPAI CHISANBOP ™ NUMBER STAIRS

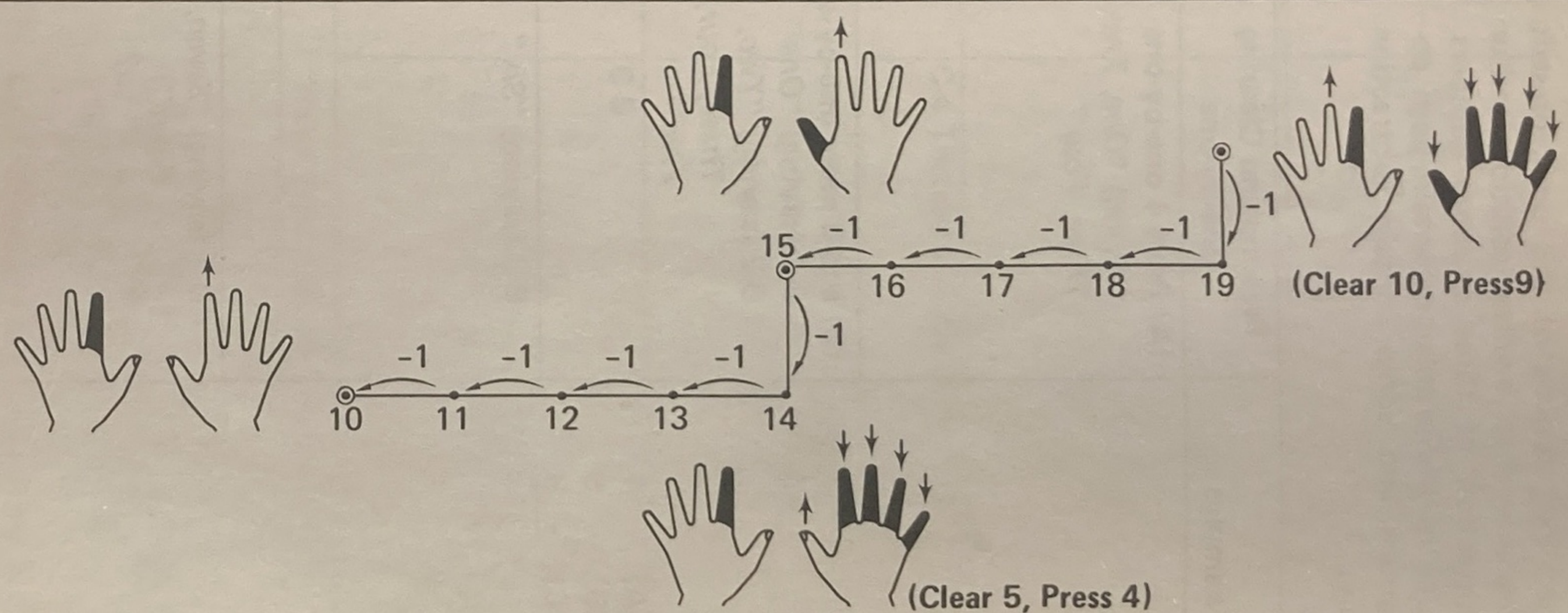
The main purpose of these stairs is to present a visual display of where the key manipulations of Chisanbop ™ subtraction occur. If students have difficulty with the key manipulations, show them a display such as this when they are learning these key manipulations. Each time they descend one level, they must make a key manipulation. They also point to the importance of the number 5 in our counting system.






These stairs feature the relationship between the numbers 5, 15, and 25 and also the relationship between the numbers 4, 14, and 24. Be careful that even though the stairs have been discontinued at three points, your students understand where they should be connected. The circular points one unit above the 9 and 19, should be conceptually joined with the circular points at 10 and 20.

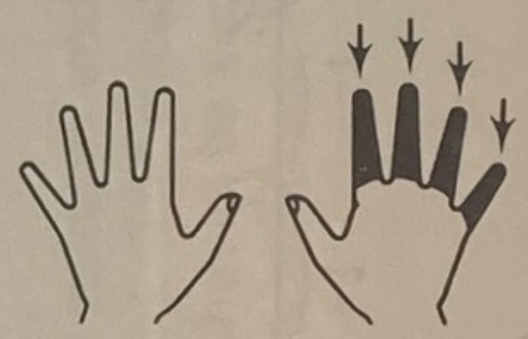
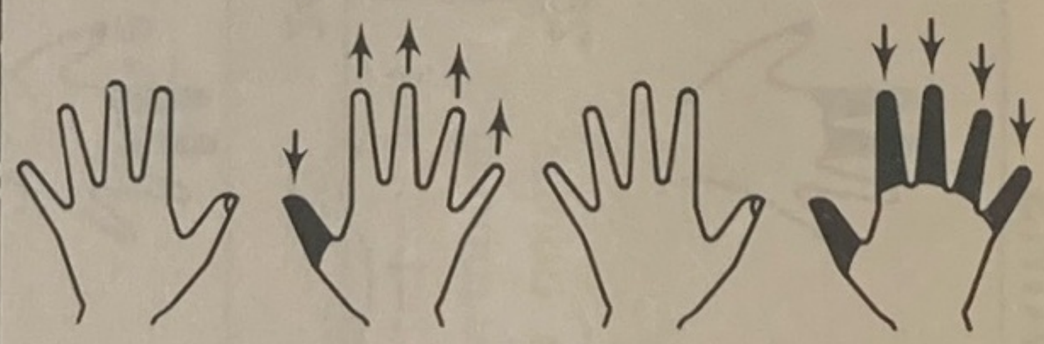
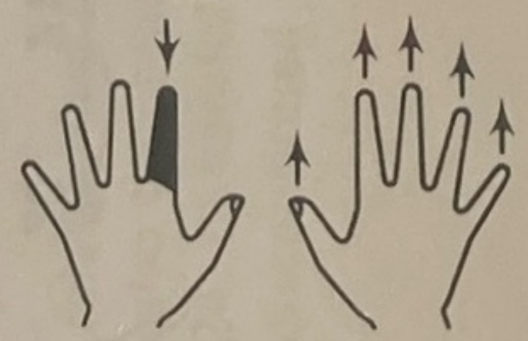
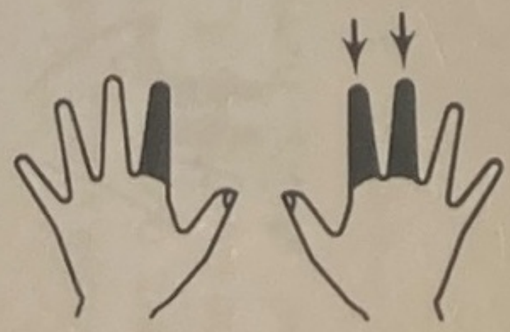

-1 -1 -1 -1 -1
25 26 27 28 29
(Clear 10, Press 9)
-1 -1 -1 -1 -1
20 21 22 23 24
(Clear 5, Press 4)

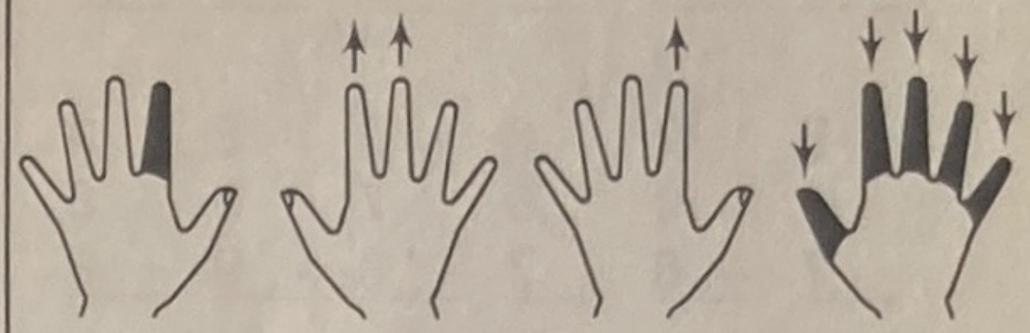
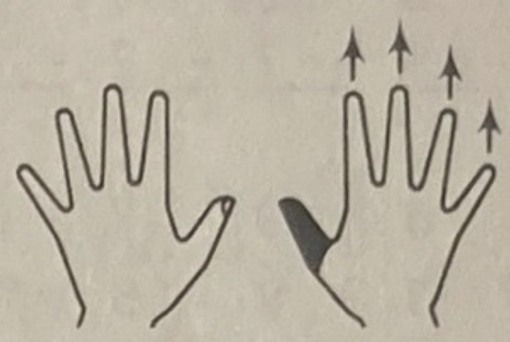
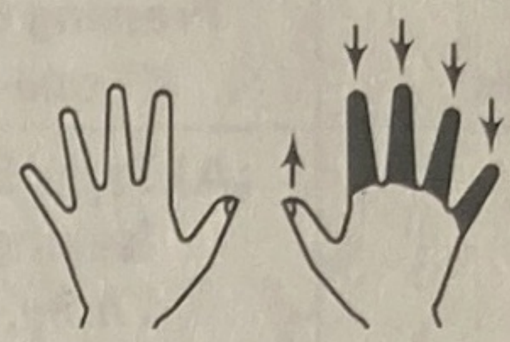
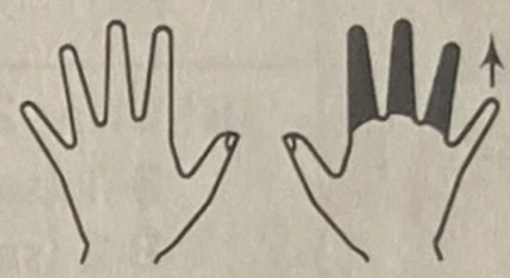

-1 -1 -1 -1 -1
15 16 17 18 19
(Clear 10, Press9)
-1 -1 -1 -1 -1
10 11 12 13 14
(Clear 5, Press 4)

## Multiple Operations

The next sections of this book present problems that involve addition and/or subtraction of three *or more* numbers. Some of the problems contain both operations and some contain subtraction only.

Multiple operations are performed in the same manner as a single operation, pressing for addition and clearing for subtraction. Study the example problems in each section. Then go over them completely before attempting the practice exercises.

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 4<br>(B) 8<br>(C) − 9<br>□ | **(A) Press 4 one-by-one** (saying) *"One, Two, Three, Four"* | **(A)**  |
| | **(B) and 8 more one-by-one**<br>**B-1** (saying) *"One"*<br>**B-2** (saying) *"Two, Three, Four, Five"* | **B-1** **B-2**  |
| | **B-3** (saying) *"Six"* | **B-3**  |
| | **B-4** (saying) *"Seven, Eight"* | **B-4**  |

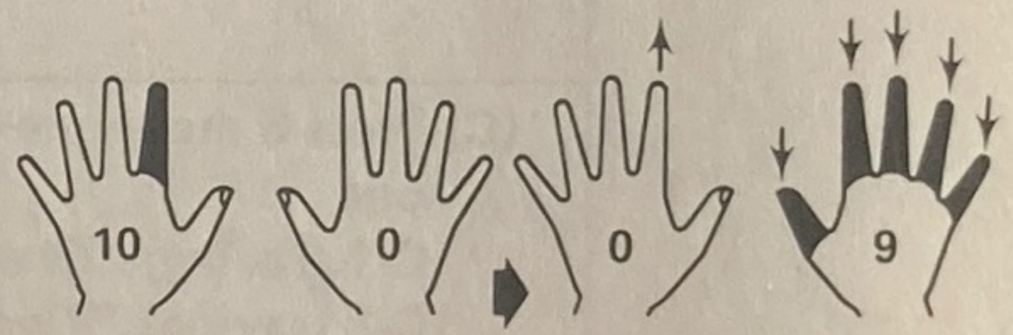
| | |
|---|---|
| **(C) Clear 9 one-by-one**<br>**C-1** (saying) *"One, Two"*<br>**C-2** (saying) *"Three"** | **C-1** **C-2** Through: (Clear 10, Press 9)  |
| **C-3** (saying) *"Four, Five, Six, Seven"* | **C-3**  |
| **C-4** (saying) *"Eight"** | **C-4** Through: (Clear 5, Press 4)  |
| **C-5** (saying) *"Nine"* | **C-5**  |

**(A)** 4
**(B)** 8
**(C)** − 9
□

*"Read Your Fingers"* **Answer:** *"3"*

## PRACTICE EXERCISES

Press the 1st addend directly. Press the 2nd addend using the highest level manipulations that the students have learned. Clear the subtrahend, one-by-one. Emphasize how + means press and − means clear.




**Clear 10 and Press 9 at the Same Time**

| | | | | | |
|---|---|---|---|---|---|
| 7 | 8 | 9 | 6 | 7 | 8 |
| 7 | 6 | 5 | 9 | 8 | 7 |
| - 5 | - 6 | - 7 | - 8 | - 9 | - 6 |

| | | | | | |
|---|---|---|---|---|---|
| 9 | 7 | 8 | 9 | 8 | 9 |
| 6 | 9 | 8 | 7 | 9 | 8 |
| - 7 | - 8 | - 7 | - 8 | - 9 | - 8 |

| | | | |
|---|---|---|---|
| 9 | 9 | 9 | 9 |
| 9 | 9 | 9 | 9 |
| - 9 | - 7 | - 2 | - 8 |

6 + 9 - 7 = □

7 + 8 - 6 = □

9 + 7 - 8 = □

8 + 7 - 9 = □

6 + 8 - 6 = □

5 + 9 - 7 = □

9 + 9 - 2 = □

9 + 9 - 8 = □

8 + 9 - 8 = □

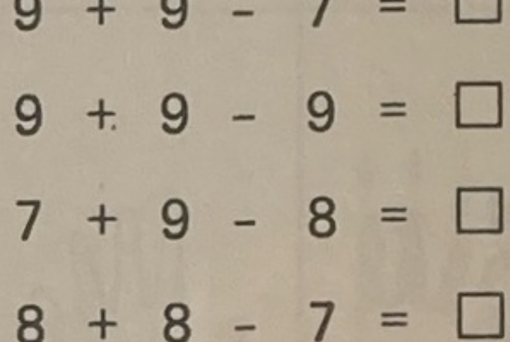
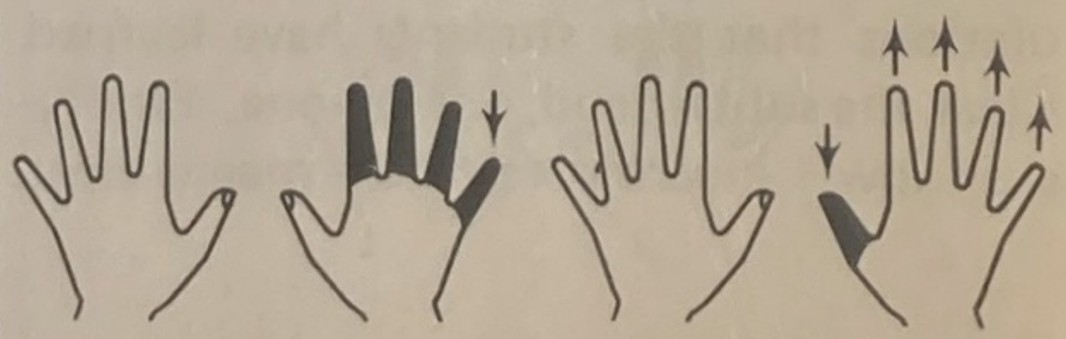
9 + 8 - 9 = □

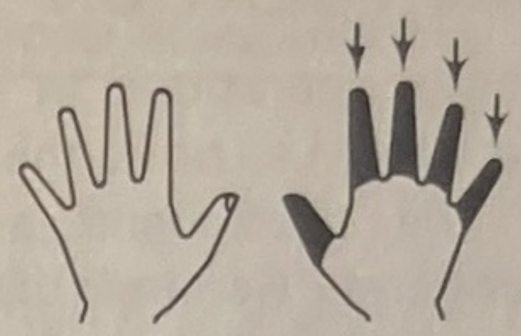
9 + 9 - 7 = □

9 + 9 - 9 = □

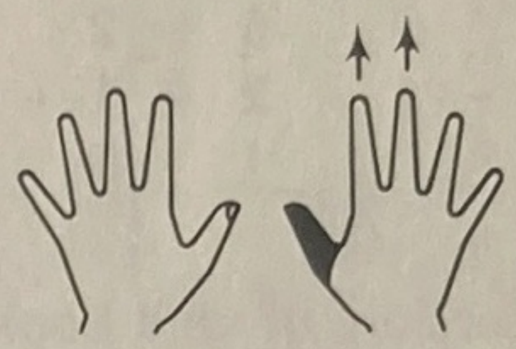
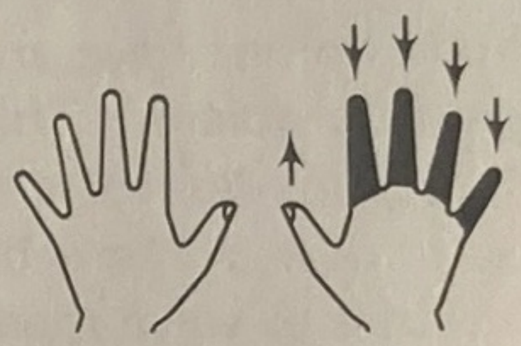
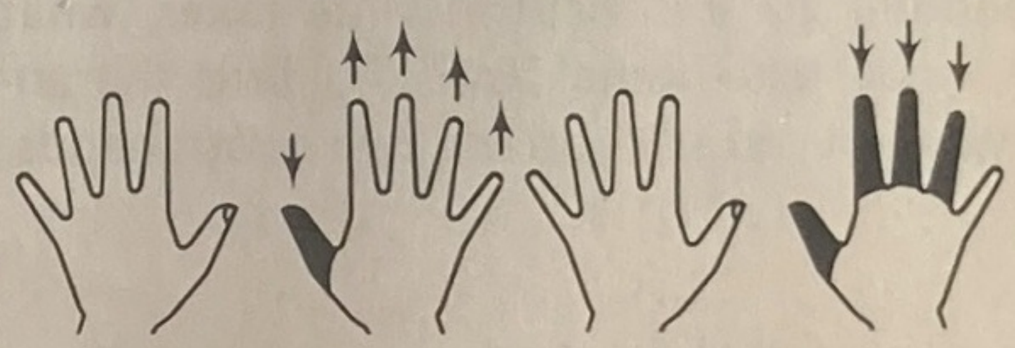
7 + 9 - 8 = □

8 + 8 - 7 = □

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 5<br>(B) - 2<br>(C) + 6<br>□ | **(A) Press 5 one-by-one**<br>(saying) *"One, Two, Three, Four, Five"* | **(A)**<br>  |
| | **(B) Clear 2 one-by-one**<br>**B-1** (saying) *"One"**<br>**B-2** (saying) *"Two"* | **B-1** Through: **B-2**<br>(Clear 5, Press 4)<br> |
| | **(C) Press 6 more one-by-one**<br>**C-1** (saying) *"One"*<br>**C-2** (saying) *"Two"* | **C-1** **C-2**<br>  |

| | |
|---|---|
| **C-3** (saying) *"Three, Four, Five, Six"* | **C-3**  |

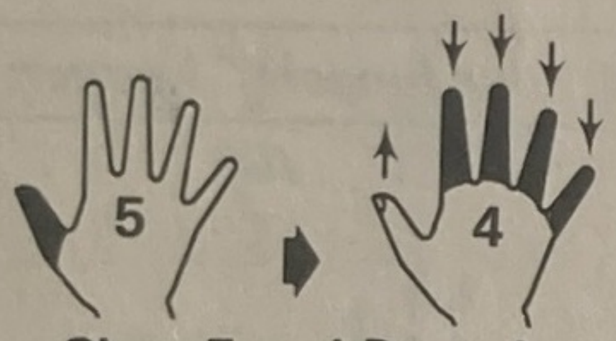
*"Read Your Fingers"* **Answer:** *"9"*

| | | |
|---|---|---|
| (A) 7<br>(B) - 3<br>(C) + 4<br>□ | **(A) Press 7 one-by-one**<br>**A-1** (saying) *"One, Two, Three, Four, Five"*<br>**A-2** (saying) *"Six, Seven"* | **A-1** **A-2** |
| | **(B) Clear 3 one-by-one**<br>**B-1** (saying) *"One, Two"* | **B 1**  |
| | **B-2** (saying) *"Three"** | **B-2** Through: (Clear 5, Press 4)  |
| | **(C) Press 4 more one-by-one**<br>**C-1** (saying) *"One"*<br>**C-2** (saying) *"Two, Three, Four"* | **C-1** **C-2**  |

*"Read Your Fingers"* **Answer:** *"8"*

## PRACTICE EXERCISES
## 5 TO 4 MANIPULATION

Perform these operations in the order that they appear, i.e. press the minuend, then clear the subtrahend and finally press the addend. Remind the students that a – sign means that they must clear.




**Clear 5 and Press 4 at the Same Time**

| | | | | | |
|---|---|---|---|---|---|
| 8 | 7 | 6 | 4 | 4 | 3 |
| - 4 | - 3 | - 2 | - 1 | - 4 | - 3 |
| 1 | 2 | 3 | 5 | 5 | 6 |

| | | | | | |
|---|---|---|---|---|---|
| 2 | 1 | 7 | 6 | 2 | 3 |
| - 2 | - 1 | - 1 | - 2 | - 1 | - 2 |
| 7 | 8 | 0 | 1 | 5 | 4 |

| | | | | | |
|---|---|---|---|---|---|
| 4 | 2 | 1 | 7 | | |
| - 1 | - 2 | - 1 | - 2 | | |
| 3 | 5 | 6 | 0 | | |

6 - 3 + 3 = □

7 - 3 + 2 = □

3 - 2 + 6 = □

2 - 2 + 7 = □

8 - 1 + 1 = □

5 - 1 + 4 = □

5 - 2 + 2 = □

6 - 2 + 1 = □

7 - 1 + 0 = □

3 - 1 + 4 = □

6 - 1 + 1 = □

5 - 1 + 2 = □

4 - 2 + 3 = □

7 - 2 + 0 = □

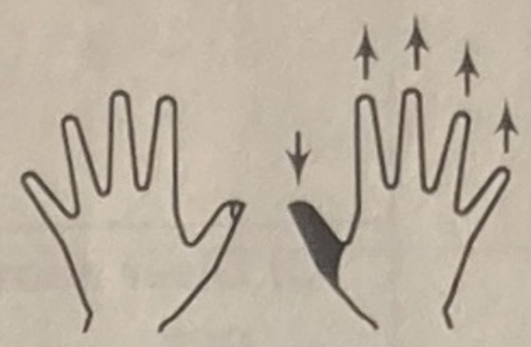
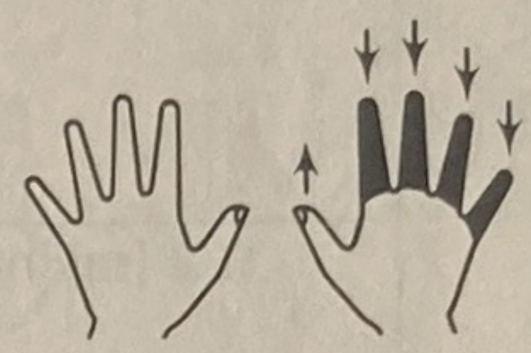
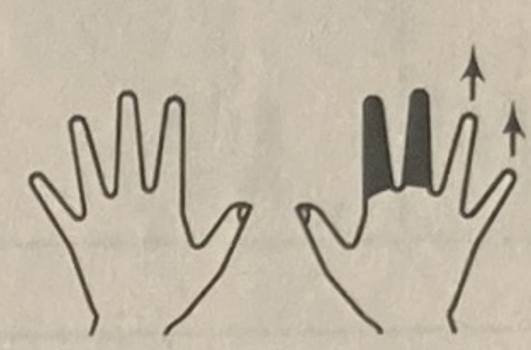
The next section of this book contains problems with 2 subtrahends. This type of problem is frequently missing from traditional math books because conventional methods do not have an easy way of handling such problems. This does not mean such problems do not arise in practical life. Up to now, we have had to take roundabout routes to solve them.

For instance, if you earn $200 a week and have $22 withheld in Federal taxes and another $7 withheld in state taxes, what is your take-home pay? To find the answer you must subtract two subtrahends.

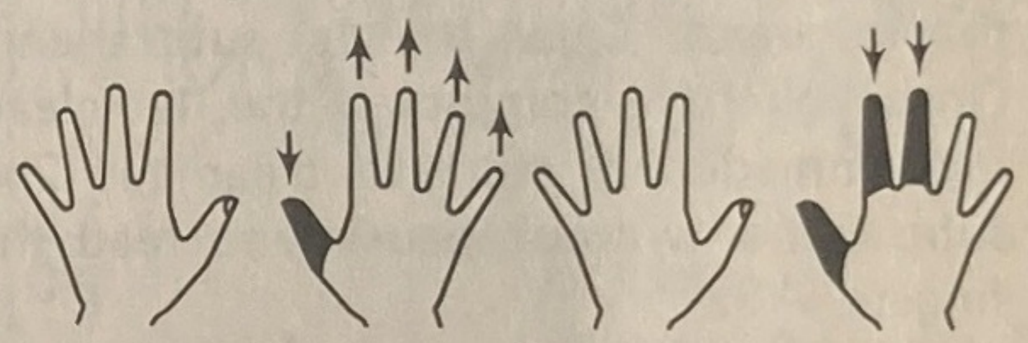
A similar problem would be if Tom had 8 apples and gave 2 to Dick and 3 to Maria. How many would Tom have left?

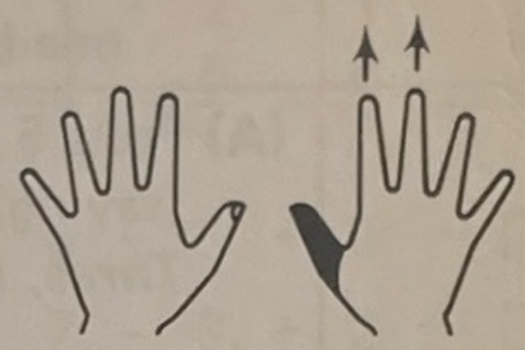
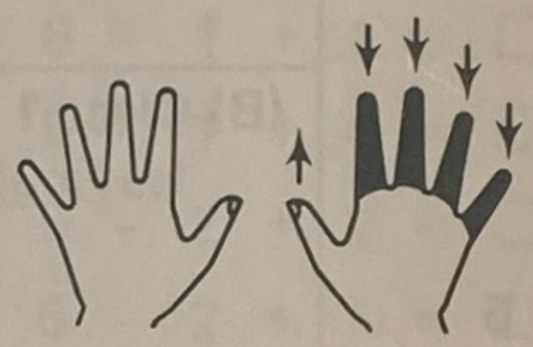
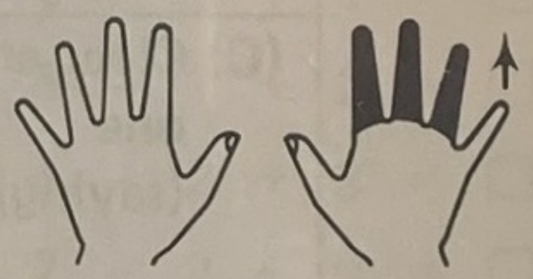
To proceed with multiple subtrahends in Chisanbop™, all you do is press the minuend and clear the subtrahends one-by-one.

Practicing problems of this nature will be of great benefit for the young student. It will begin to give him some insight into the nature of negative numbers. e.g. $9 - 2 - 4 = 9 - 6 = 9 - (2 + 4)$, and thus $(-2) + (-4) = -6$.

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 5<br>(B) - 1<br>(C) - 2<br>□ | **(A) Press 5 one-by-one**<br>(saying) *"One, Two, Three, Four, Five"* | **(A)**<br>  |
| | **(B) Clear 1**<br>(saying) *"One"** | **(B)** Through: (Clear 5, Press 4)<br>  |
| | **(C) Clear another 2 one-by-one**<br>(saying) *"One, Two"* | **(C)**<br>  |

*"Read Your Fingers"* **Answer: *"2"***

| | | |
|---|---|---|
| (A) 7<br>(B) - 2<br>(C) - 2<br>□ | **(A) Press 7 one-by-one**<br>**A-1** (saying) *"One, Two, Three, Four, Five"*<br>**A-2** (saying) *"Six, Seven"* | **A-1** **A-2**<br>  |

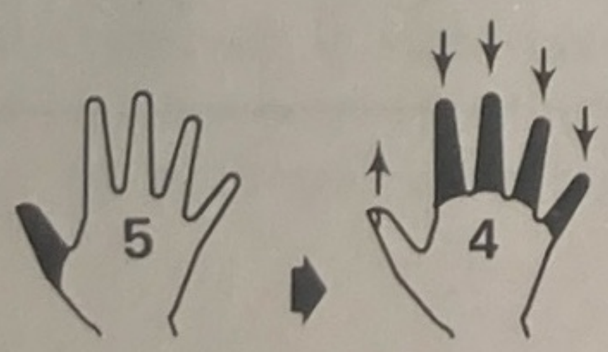
| | |
|---|---|
| **(B) Clear 2 one-by-one** (saying) *"One, Two"* | **(B)**  |
| **(C) Clear another 2 one-by-one**<br>**C-1** (saying) *"One"** | **C-1** Through: (Clear 5, Press 4)  |
| **C-2** (saying) *"Two"* | **C-2**  |

(A) 7
(B) - 2
(C) - 2
□

*"Read Your Fingers"* **Answer:** *"3"*

## PRACTICE EXERCISES

Chisanbop ™ eliminates much of the intermediate work of arithmetic. Press the minuend. Clear the 1st subtrahend. Once you have completed the 1st clearing, immediately start to clear the 2nd subtrahend without pausing to read the fingers.




**Clear 5 and Press 4 at the Same Time**

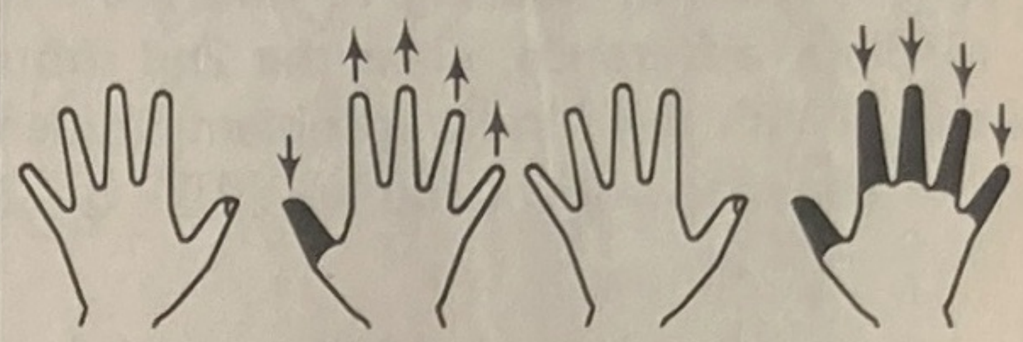
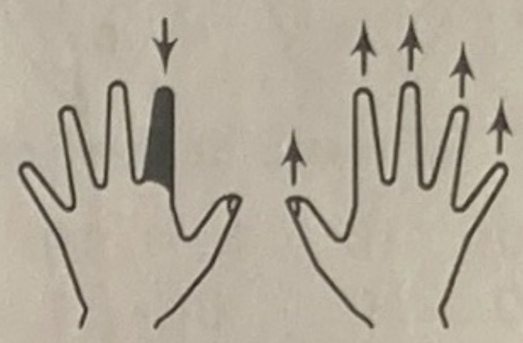
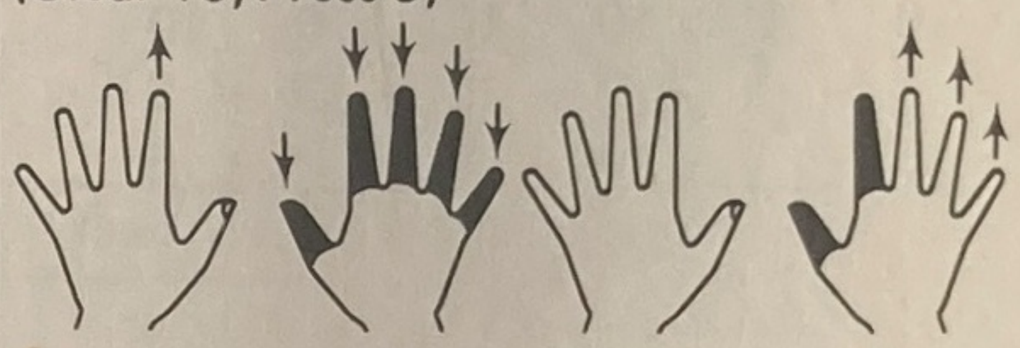
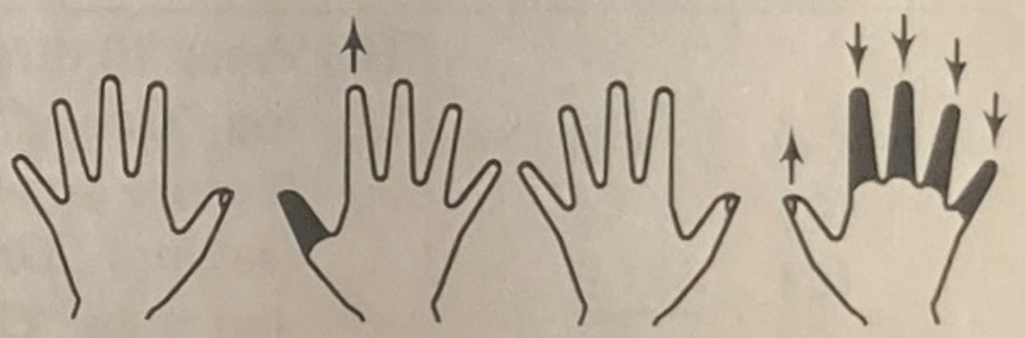
| | | | | |
|---|---|---|---|---|
| 7 | 7 | 7 | 7 | 7 |
| - 1 | - 1 | - 2 | - 2 | - 2 |
| - 5 | - 6 | - 1 | - 2 | - 3 |

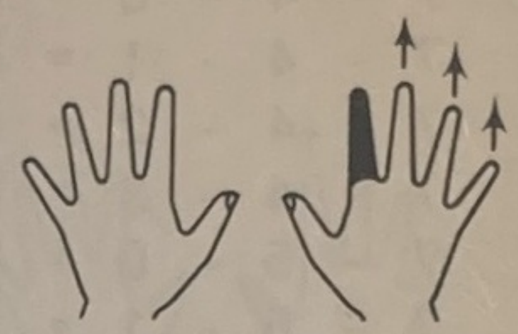
| | | | | |
|---|---|---|---|---|
| 7 | 7 | 7 | 8 | 8 |
| - 2 | - 2 | - 2 | - 0 | - 0 |
| - 4 | - 5 | - 0 | - 1 | - 2 |

| | | | | |
|---|---|---|---|---|
| 8 | 8 | 8 | 8 | 8 |
| - 1 | - 1 | - 1 | - 1 | - 0 |
| - 0 | - 1 | - 2 | - 3 | - 3 |

7 - 4 - 0 = ☐
7 - 4 - 1 = ☐
7 - 4 - 2 = ☐
7 - 4 - 3 = ☐
7 - 5 - 0 = ☐
7 - 5 - 1 = ☐
8 - 2 - 0 = ☐

8 - 2 - 1 = ☐
8 - 2 - 2 = ☐
8 - 2 - 3 = ☐
8 - 3 - 0 = ☐
8 - 3 - 1 = ☐
8 - 3 - 2 = ☐
8 - 3 - 3 = ☐

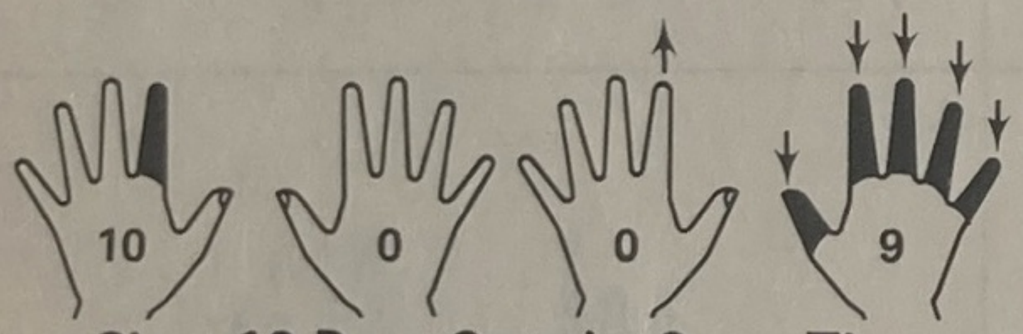
| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 10<br>(B) - 4<br>(C) - 5<br>☐ | (A) Press 10 directly (saying "Onety") or one-by-one<br>A-1 (saying) *"One, Two, Three, Four, Five"*<br>A-2 (saying) *"Six, Seven, Eight, Nine"* | A-1 A-2  |
| | A-3 (saying) *"Onety"* (10) | A-3  |
| | (B) Clear 4 one-by-one<br>B-1 (saying) *"One"**<br>B-2 (saying) *"Two, Three, Four"* | B-1 Through: (Clear 10, Press 9) B-2  |
| | (C) Clear another 5 one-by-one<br>C-1 (saying) *"One"*<br>C-2 (saying) *"Two"** | C-1 C-2 Through: (Clear 5, Press 4)  |

| | |
|---|---|
| **C-3** (saying) *"Three, Four, Five"* | **C-3**  |

*"Read Your Fingers"* **Answer: *"1"***

## PRACTICE EXERCISES

Press the minuend. Clear the 1st subtrahend. Without pausing to find the intermediate difference, clear the 2nd subtrahend. With this kind of problem, students will begin to understand how 10 - 5 - 2 = 10 - (5 + 2).




**Clear 10 Press 9 at the Same Time**

| | | | | | |
|---|---|---|---|---|---|
| 10 | 10 | 10 | 10 | 10 | 10 |
| - 0 | - 0 | - 0 | - 0 | - 1 | - 1 |
| - 1 | - 4 | - 3 | - 2 | - 0 | - 2 |

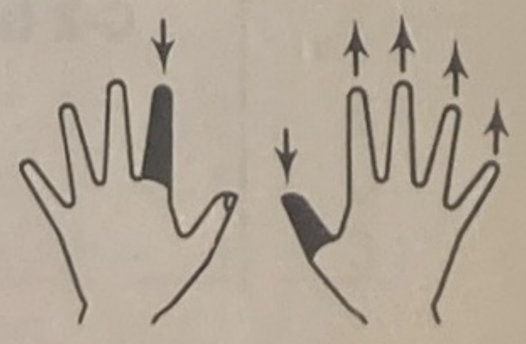
| | | | | | |
|---|---|---|---|---|---|
| 10 | 10 | 10 | 10 | 10 | 10 |
| - 1 | - 1 | - 2 | - 2 | - 2 | - 2 |
| - 3 | - 1 | - 6 | - 8 | - 7 | - 5 |

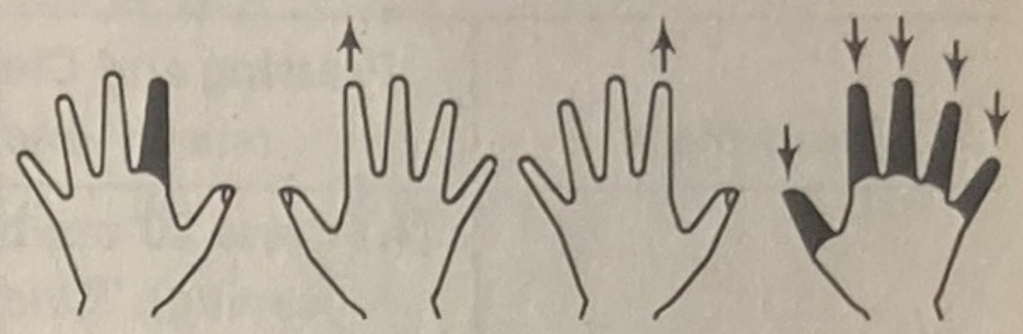
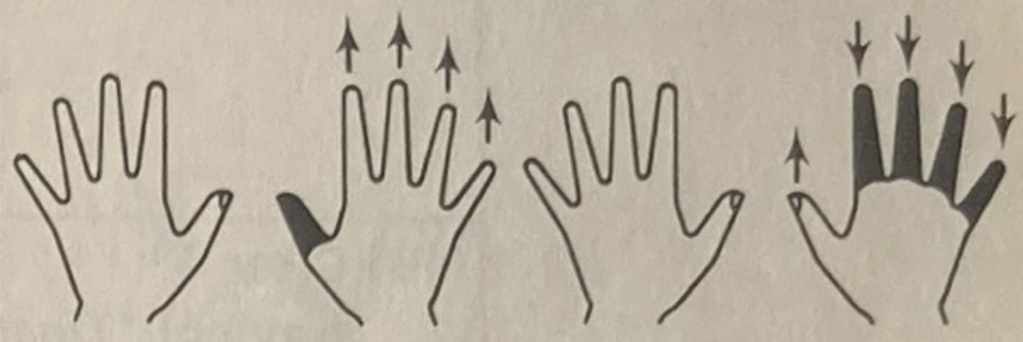
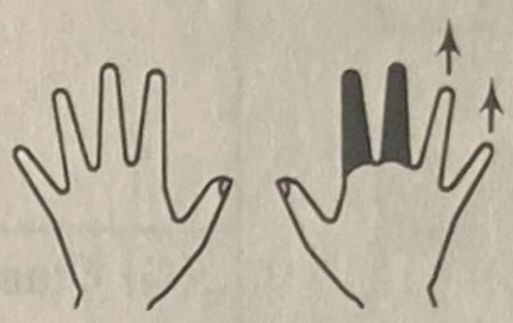
| | | | | |
|---|---|---|---|---|
| 10 | 10 | 10 | 10 | 10 |
| - 7 | - 7 | - 7 | - 8 | - 3 |
| - 1 | - 2 | - 3 | - 1 | - 7 |

| | | | | |
|---|---|---|---|---|
| 10 | 10 | 10 | 10 | 10 |
| - 3 | - 3 | - 3 | - 4 | - 4 |
| - 5 | - 1 | - 6 | - 1 | - 3 |

| | | | | |
|---|---|---|---|---|
| 10 | 10 | 10 | 10 | 10 |
| - 4 | - 4 | - 5 | - 5 | - 5 |
| - 2 | - 4 | - 4 | - 3 | - 2 |

| | | | | |
|---|---|---|---|---|
| 10 | 10 | 10 | 10 | 10 |
| - 5 | - 6 | - 6 | - 6 | - 6 |
| - 5 | - 1 | - 2 | - 3 | - 0 |

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 15<br>(B) - 4<br>(C) - 9<br>□ | (A) **Press 10 directly (saying "Onety") and then 5 more one-by-one** (saying) *"Onety"* (10) and then *"One, Two, Three, Four, Five"* | (A)  |

| | |
|---|---|
| **(B) Clear 4 one-by-one**<br>**B-1** (saying) *"One"**<br>**B-2** (saying) *"Two, Three, Four"* | **B-1** Through: (Clear 5, Press 4) **B-2**<br> |
| **(C) Clear another 9 one-by-one**<br>**C-1** (saying) *"One"*<br>**C-2** (saying) *"Two"** | **C-1** **C-2** Through: (Clear 10, Press 9)<br>  |
| **C-3** (saying) *"Three, Four, Five, Six"*<br>**C-4** (saying) *"Seven"** | **C-3** **C-4** Through: (Clear 5, Press 4)<br>  |
| **C-5** (saying) *"Eight, Nine"* | **C-5**<br>  |

(A) 15
(B) - 4
(C) - 9
□

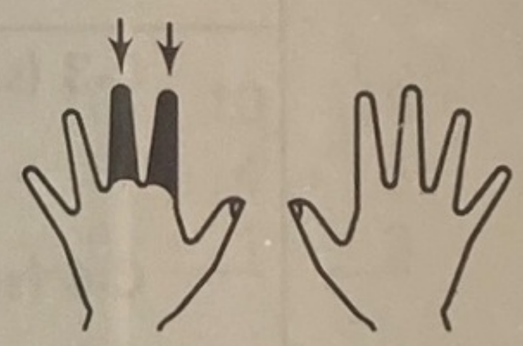
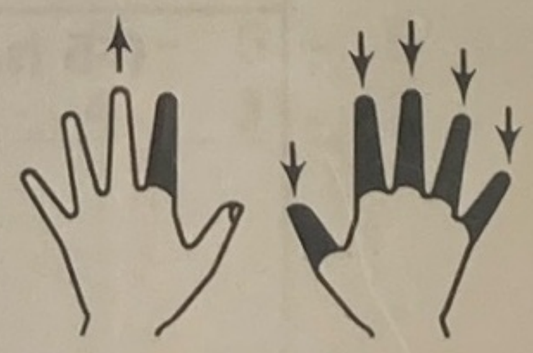
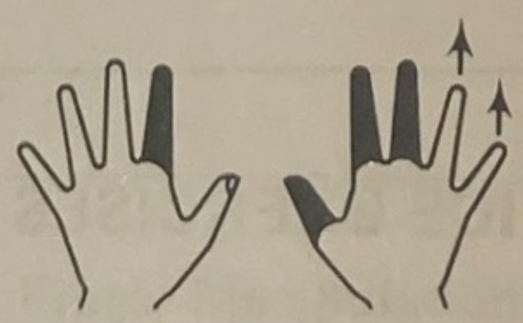
*"Read Your Fingers"* **Answer:** *"2"*

## PRACTICE EXERCISES

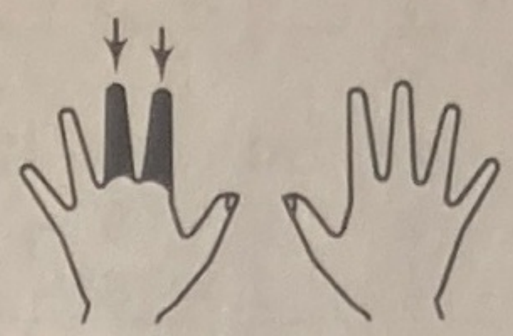
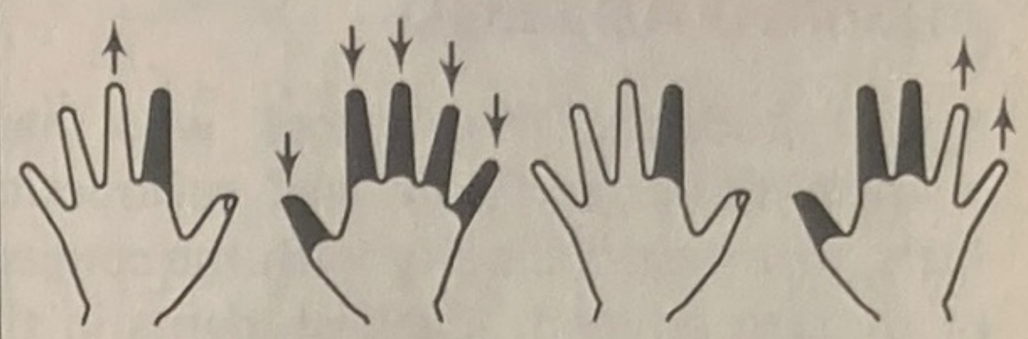
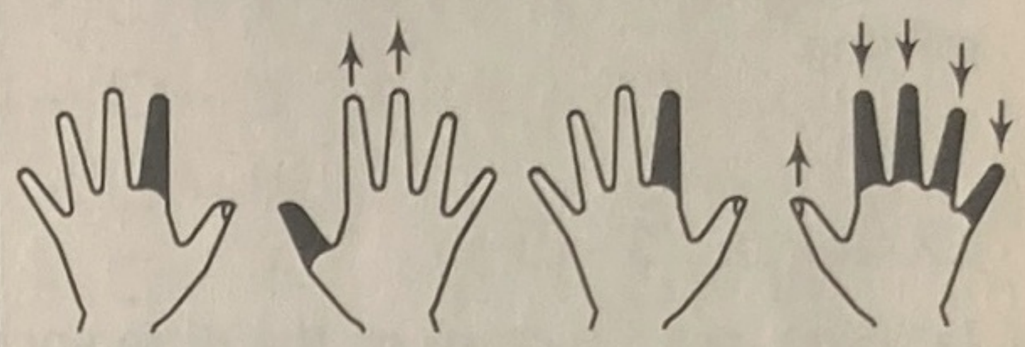
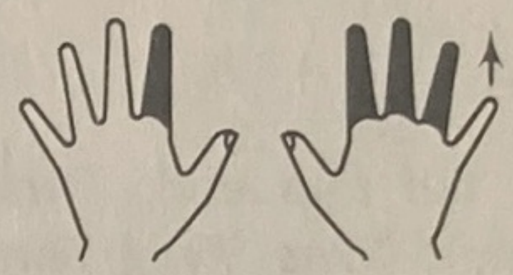
Press both digits of the minuend. This is not columnar subtraction. Clear both subtrahends and then read you fingers. These problems contain examples of both manipulations.

| | | | | |
|---|---|---|---|---|
| 15 | 15 | 15 | 15 | 15 |
| - 0 | - 0 | - 0 | - 0 | - 1 |
| - 6 | - 9 | - 7 | - 8 | - 6 |
| 15 | 15 | 15 | 15 | 15 |
| - 1 | - 3 | - 3 | - 2 | - 2 |
| - 0 | - 4 | - 1 | - 9 | - 1 |

```
 15   15   15   15   15   15
- 2  - 2  - 1  - 1  - 3  - 3
- 0  - 6  - 2  - 8  - 3  - 2

 15   15   15   15   15   15
- 3  - 4  - 4  - 4  - 4  - 3
- 5  - 4  - 9  - 3  - 8  - 7

 15   15   15   15
- 3  - 5  - 5  - 5
- 6  - 1  - 9  - 3

 15   15   15   15
- 6  - 6  - 6  - 6
- 9  - 5  - 3  - 7
```

| Examples | Pressing and Clearing one-by-one | Finger Manipulation |
|---|---|---|
| (A) 20<br>(B) - 1<br>(C) - 2 | **(A) Press 20 ten-by-ten** (saying) *"Onety* (10), *Twenty"* | **(A)**  |
| | **(B) Clear 1** (saying) *"One"** | **(B)** Through: (Clear 10, Press 9)  |
| | **(C) Clear another 2 one-by-one** (saying) *"One, Two"* | **(C)**  |

*"Read Your Fingers"* **Answer:** *"17"*

| | |
|---|---|
| **(A) Press 20 ten-by-ten** (saying) *"Onety* (10), *Twenty"* | **(A)**  |
| **(B) Clear 3 one-by-one**<br>**B-1** (saying) *"One"**<br>**B-2** (saying) *"Two, Three"* | **B-1** Through: (Clear 10, Press 9) **B-2**  |
| **(C) Clear another 4 one-by-one**<br>**C-1** (saying) *"One, Two"*<br>**C-2** (saying) *"Three"** | **C-1** **C-2** Through: (Clear 5, Press 4)  |
| **C-3** (saying) *"Four"* | **C-3**  |

(A) 20
(B) - 3
(C) - 4

*"Read Your Fingers"* **Answer:** *"13"*

## PRACTICE EXERCISES

### 5 TO 4 & 10 TO 9 MANIPULATIONS

Press the minuend directly or ten-by-ten. Clear both subtrahends, one-by-one. Make sure that the 1st time you Clear 10 Press 9 that it is the 20th finger that you clear.

| | | | | |
|---|---|---|---|---|
| 20 | 20 | 20 | 20 | 20 |
| - 0 | - 0 | - 0 | - 0 | - 1 |
| - 7 | - 6 | - 9 | - 8 | - 1 |
| 20 | 20 | 20 | 20 | 20 |
| - 1 | - 1 | - 1 | - 2 | - 2 |
| - 4 | - 3 | - 8 | - 9 | - 6 |

| | | | | |
|---|---|---|---|---|
| 20 | 20 | 20 | 20 | 20 |
| - 2 | - 2 | - 3 | - 3 | - 3 |
| - 4 | - 7 | - 8 | - 5 | - 7 |

| | | | | | |
|---|---|---|---|---|---|
| 20 | 20 | 20 | 20 | 20 | 20 |
| - 8 | - 8 | - 8 | - 8 | - 7 | - 7 |
| - 5 | - 7 | - 3 | - 9 | - 4 | - 2 |

| | | | | |
|---|---|---|---|---|
| 20 | 20 | 20 | 20 | 20 |
| - 3 | - 9 | - 9 | - 9 | - 9 |
| - 4 | - 4 | - 2 | - 8 | - 6 |

| | | | | | |
|---|---|---|---|---|---|
| 20 | 20 | 20 | 20 | 20 | 20 |
| - 7 | - 7 | - 6 | - 6 | - 6 | - 6 |
| - 8 | - 6 | - 5 | - 7 | - 3 | - 9 |

## MISSING ADDEND

Many students, even those who have learned basic addition and subtraction facts, manifest difficulty with the concept of missing addend. Even students in the fourth grade and above sometimes become confused when they must work with arithmetic statements where numbers are missing.

To solve missing addend examples using Stage 1 Subtraction, the only requirement is that the student must be able to manipulate the fingers correctly and count to the picture of the difference. The process is primarily visual and tactile. The examples that follow illustrate how the process works.

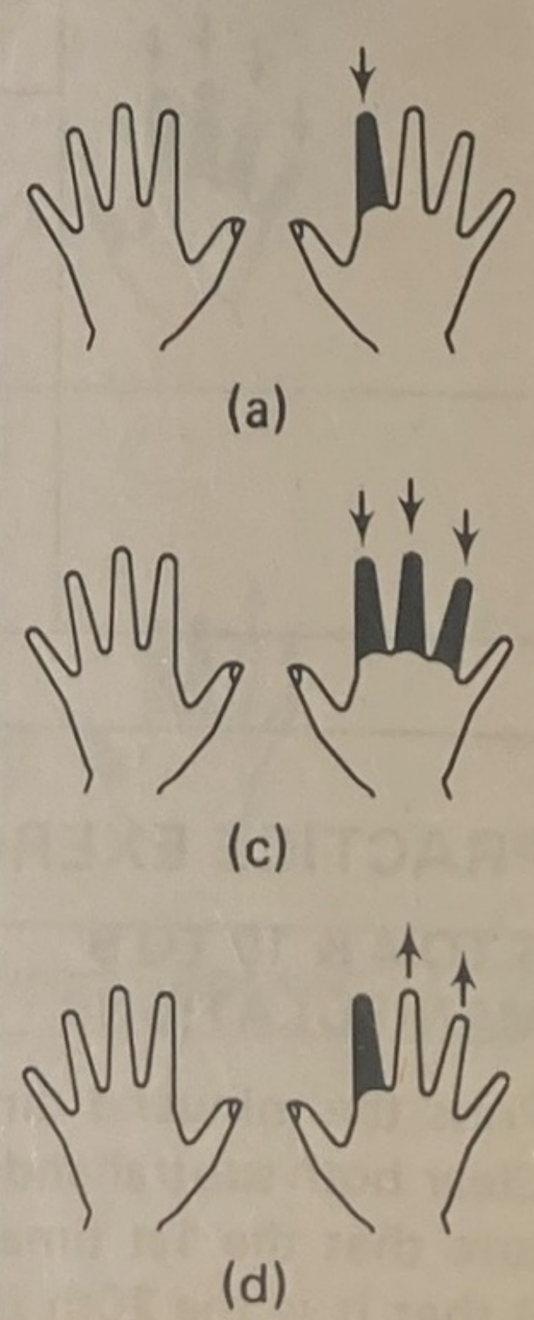
EXAMPLE: $3 - \square = 1$

(a) First, get a picture of the difference on your fingers, *One* (1) in this example.

(b) Clear your fingers, and remember the picture you just made.

(c) Press the minuend, *Three* (3) in this example, one-by-one (saying): *"One, Two, Three"*

(d) Now Clear (one-by-one) from that number, until your fingers reach the picture of the difference you are remembering. Make sure to count as you clear. You thus will count: "one, two."



Note: The visual sense is used in making sure that the pressed fingers are the same in steps A and D. The oral counting determines the missing addend. The procedure is extremely beneficial in so far as the number missing in this subtraction problem is found by clearing. Once the students are familiar with the concept of missing addends in subtraction problems, they can skip Steps A and B. The procedure is to press the minuend and then to clear until the fingers show the number in the difference. By counting as they clear they find their answer.

## PRACTICE EXERCISES ALGEBRA

Clear unit-by-unit until a picture of the difference is reached on your fingers. When first learning this method, it is better to use many problems with the same differences. In this way, the students do not have to concentrate on remembering the difference but on learning how to proceed.

Once the students are comfortable with the procedure, you can begin to give them problems with a variety of differences.

1 - □ = 1
2 - □ = 1
3 - □ = 1
4 - □ = 1
5 - □ = 1
6 - □ = 1
7 - □ = 1
8 - □ = 1
9 - □ = 1
2 - □ = 2
3 - □ = 2
4 - □ = 2
6 - □ = 2
6 - □ = 1
6 - □ = 0
7 - □ = 4
7 - □ = 3
7 - □ = 2
7 - □ = 1
7 - □ = 0
8 - □ = 4
8 - □ = 3
8 - □ = 0
9 - □ = 1

5 - □ = 2
6 - □ = 2
7 - □ = 2
8 - □ = 2
9 - □ = 2
3 - □ = 2
4 - □ = 2
5 - □ = 2
6 - □ = 2
7 - □ = 2
8 - □ = 2
9 - □ = 2
6 - □ = 3
5 - □ = 2
5 - □ = 1
5 - □ = 5
6 - □ = 4
5 - □ = 3
5 - □ = 0
5 - □ = 4
9 - □ = 4
8 - □ = 2
9 - □ = 0
9 - □ = 3

## MISSING ADDEND

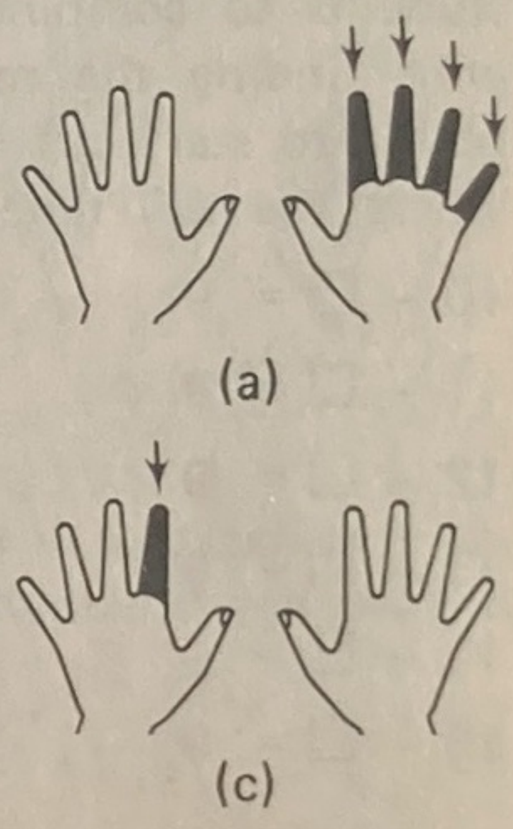
EXAMPLE: 10 - □ = 4

(a) First, get a picture of the difference on your fingers, *Four* (4) in this example.

(b) Clear your fingers and remember the picture you just made.

(c) Press the minuend, *Ten* (10) in this example, pressing 10 directly (saying): *"Onety"*

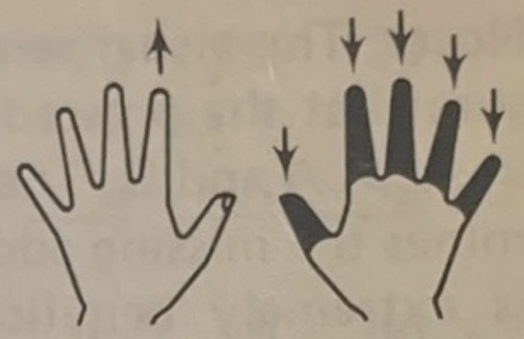
(d) Now Clear (one-by-one) from the number (10) until your fingers reach a picture of the difference (4).



(a)

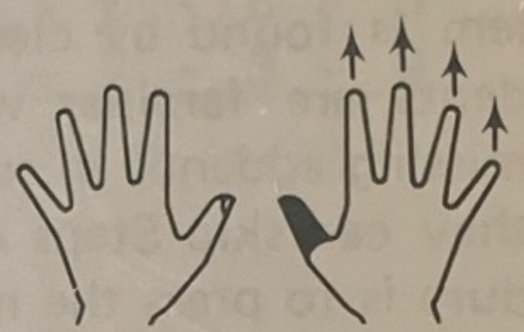
(c)

(e-1) Clear (saying): *"ONE"* (Through: Clear 10, Press 9)



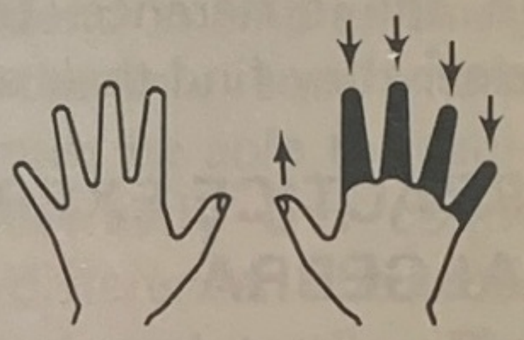
(e-1)

(e-2) Clear (saying): *"Two, Three, Four, Five"*



(e-2)

(e-3) Then Clear again (saying): *"SIX"* (Through: Clear 5, Press 4) After you say *"Six,"* your fingers should look like illustration (a)



(e-3)

Once the students are familiar with the concept of missing addends in a subtraction problem they can skip steps A & B. The procedure is to press the minuend and then to clear until the fingers show the number in the difference. By counting as they clear they find their answer.

## PRACTICE EXERCISES

Once again, since you are now asking the students to combine key manipulations with finding the missing addend, it is better to start out with many problems where the difference remains the same.

10 - □ = 9
11 - □ = 9
12 - □ = 9
13 - □ = 9
14 - □ = 9
15 - □ = 9

13 - □ = 8
14 - □ = 8
15 - □ = 8
16 - □ = 8
17 - □ = 8
10 - □ = 7

16 - □ = 9
17 - □ = 9
18 - □ = 9
10 - □ = 8
11 - □ = 8
12 - □ = 8

11 - □ = 7
12 - □ = 7
13 - □ = 7
14 - □ = 7
15 - □ = 7
16 - □ = 7

As the students grow more comfortable with the procedure, you challenge them by asking them to remember various differences.

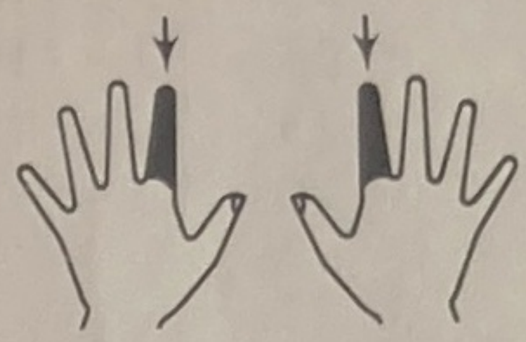
10 - □ = 4
10 - □ = 2
10 - □ = 10
10 - □ = 0
10 - □ = 7
10 - □ = 5

10 - □ = 1
10 - □ = 3
10 - □ = 6
10 - □ = 8
10 - □ = 9
10 - □ = 5

| | | | |
|---|---|---|---|
| 11 - □ = 2 | 11 - □ = 7 | 11 - □ = 9 | 11 - □ = 10 |
| 11 - □ = 5 | 11 - □ = 8 | 11 - □ = 3 | 11 - □ = 0 |
| 11 - □ = 6 | 11 - □ = 1 | 11 - □ = 4 | 11 - □ = 11 |

EXAMPLE: 20 - □ = 11

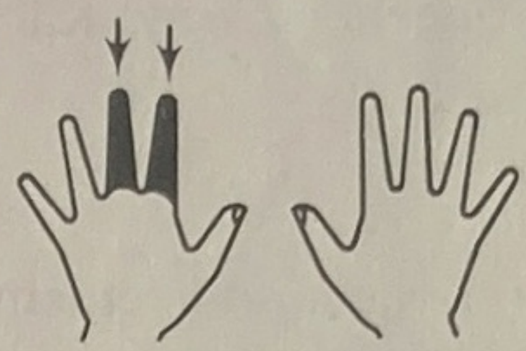
(a) First, get a picture of the difference on your fingers, *Onety-one* (11) in this example.



(a)

(b) Clear your fingers and remember the picture you just made.

(c) Press the minuend, *Twenty* (20) in this example, pressing 20 ten-by-ten (saying): *"Onety, Twenty"*



(c)

(d) Now Clear (one-by-one) from that number (20) until your fingers reach a picture of the difference (11).

(e-1) Clear (saying): *"ONE"* (Through: Clear 10, Press 9)

(e-2) Then Clear again (saying): *"Two, Three, Four, Five"*

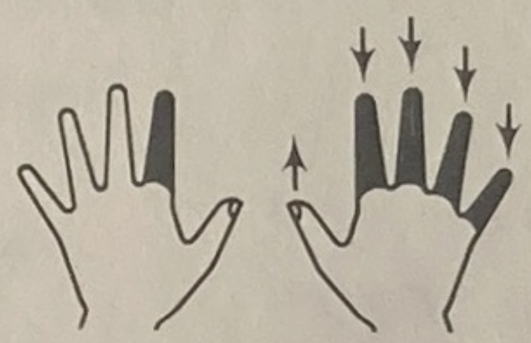
(e-3) Then Clear again (saying): *"SIX"* (Through: Clear 5, Press 4)

(e-4) Then Clear again (saying): *"Seven, Eight, Nine"* After you say *"Nine,"* your fingers should look like illustration (a)

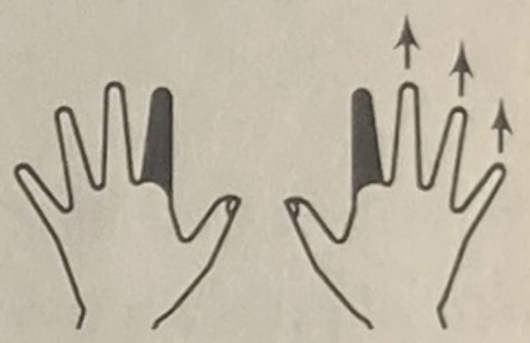
(e-1)



(e-2)



(e-3)



(e-4)

Once the students are familiar with the concept of missing addends in subtraction problems, they can skip steps A and B. The procedure is to press the minuend and then to clear until the fingers show the number in the difference.

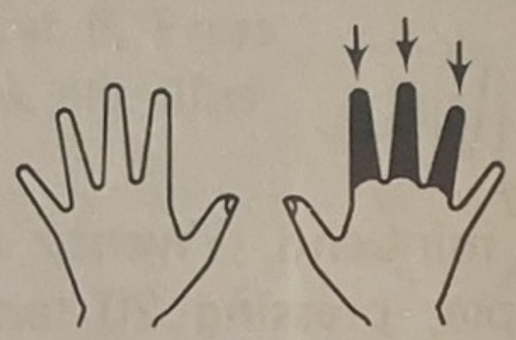
By counting as they clear they find their answer.

## PRACTICE EXERCISES

| | | | |
|---|---|---|---|
| 20 - □ = 16 | 21 - □ = 12 | 27 - □ = 19 | 25 - □ = 16 |
| 20 - □ = 12 | 22 - □ = 14 | 28 - □ = 19 | 26 - □ = 17 |
| 20 - □ = 14 | 23 - □ = 16 | 21 - □ = 16 | 27 - □ = 18 |
| 20 - □ = 17 | 24 - □ = 18 | 22 - □ = 17 | 28 - □ = 19 |
| 20 - □ = 19 | 25 - □ = 19 | 23 - □ = 18 | 21 - □ = 19 |
| 20 - □ = 13 | 26 - □ = 18 | 24 - □ = 19 | 28 - □ = 20 |

EXAMPLE: 8 + 2 - □ = 3

(a) First, get a picture of the difference on your fingers, *Three* (3) in this example.



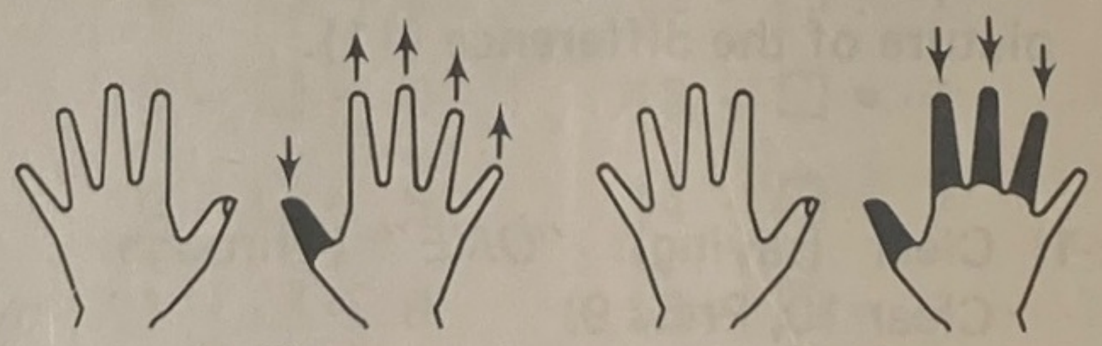
(a)

(b) Clear your fingers and remember the picture you just made.

(c) Press the first addend, *Eight* (8) in this example one-by-one.

(d-1) Press (saying): *"One, Two, Three, Four, FIVE"*

(d-2) Press again (saying): *"Six, Seven, Eight"*

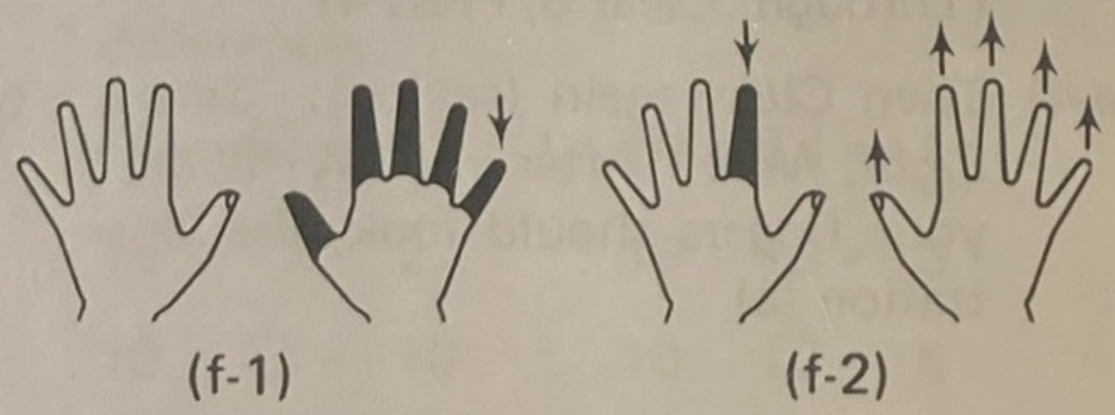


(d-1) (d-2)

(e) Keep your fingers pressed and press the next addend, *Two* (2), one-by-one.

(f-1) Press (saying): *"One"*

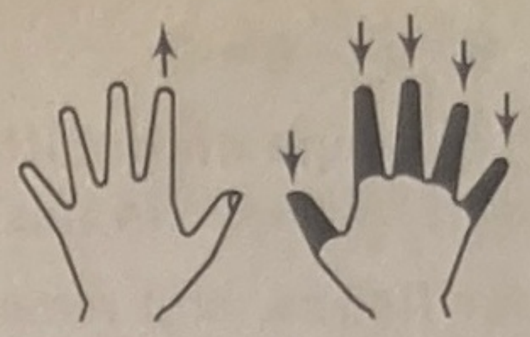
(f-2) Press again (saying): *"TWO"*



(f-1) (f-2)

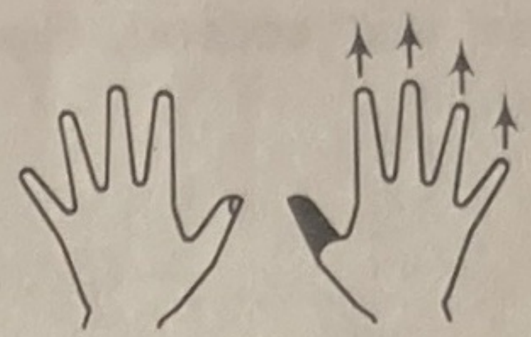
(g) Now let's Clear (one-by-one) from that number (10) until you get a picture of the difference (3) on our fingers.

(h-1) Clear (saying):
*"ONE"* (Through: Clear 10, Press 9)



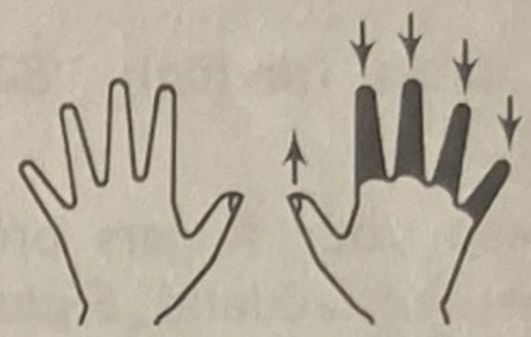
(h-1)

(h-2) Clear again (saying):
*"Two, Three, Four, Five"*



(h-2)

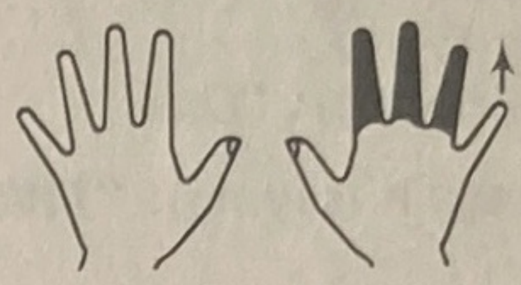
(h-3) Clear again (saying):
*"SIX"* (Through: Clear 5, Press 4)



(h-3)

(h-4) Then Clear (saying): *"Seven"*

After you say *"Seven,"* your fingers should look like illustration (a).



(h-4)

By counting to seven you have found the answer.

Once the students are familiar with the concept of missing addends in a subtraction problem they can skip steps A and B. The procedure is to press the minuend and then to clear until the fingers show the number in the difference.

By counting as they clear they find their answers.

## PRACTICE EXERCISES

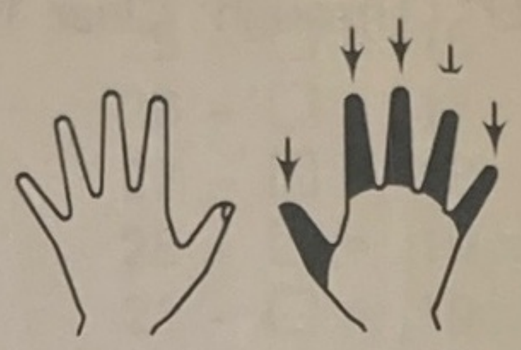
| | | | |
|---|---|---|---|
| 1 + 6 - □ = 6 | 1 + 6 - □ = 6 | 3 + 6 - □ = 6 | 2 + 7 - □ = 7 |
| 2 + 2 - □ = 2 | 3 + 4 - □ = 5 | 7 + 2 - □ = 6 | 6 + 3 - □ = 7 |
| 4 + 3 - □ = 5 | 6 + 3 - □ = 8 | 1 + 8 - □ = 8 | 1 + 7 - □ = 7 |
| 3 + 1 - □ = 1 | 6 + 0 - □ = 5 | 5 + 4 - □ = 8 | 4 + 4 - □ = 7 |
| 3 + 6 - □ = 8 | 6 + 1 - □ = 5 | 4 + 5 - □ = 5 | 5 + 3 - □ = 6 |
| 1 + 3 - □ = 3 | 4 + 0 - □ = 0 | 8 + 1 - □ = 5 | 6 + 2 - □ = 5 |

EXAMPLE: 8 - □ + 8 = 9

(a) First, get a picture of the difference on your fingers, *Nine* (9) in this example.

(b) Clear your fingers, and remember the picture you just made.

(c) Press the first addend, *Eight* (8) in this example.

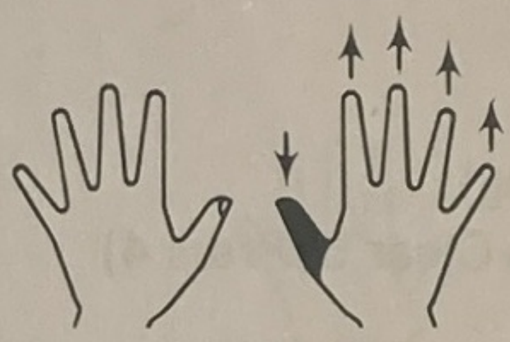


(a)

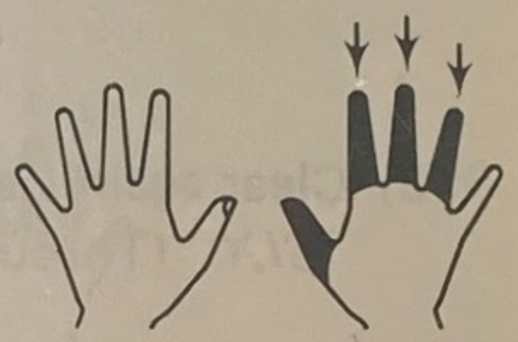
(d-1) Press (saying): *"One, Two, Three, Four, FIVE"*

(d-2) Then press (saying): *"Six, Seven, Eight"*

(e) Now keep your fingers pressed and press the next addend, *Eight* (8), one-by-one.



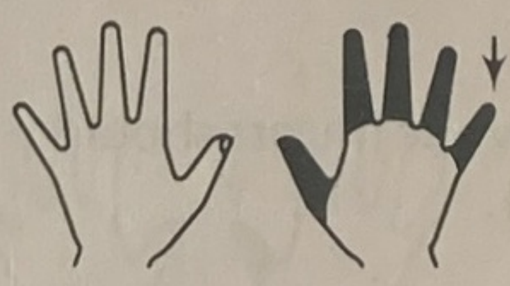
(d-1)



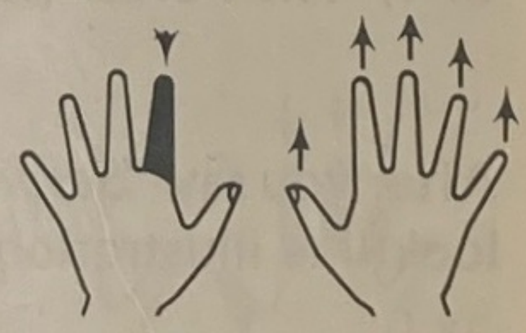
(d-2)

(f-1) Press (saying): *"One"*

(f-2) Press again (saying): *"TWO"*



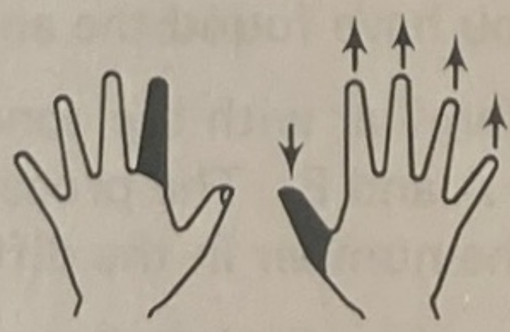
(f-1)



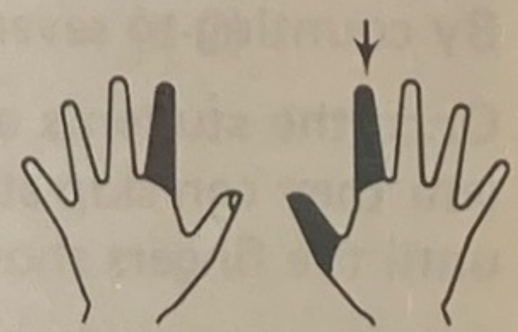
(f-2)

(f-3) Press again (saying): *"Three, Four, Five, Six, SEVEN"*

(f-4) Press again (saying): *"Eight"*



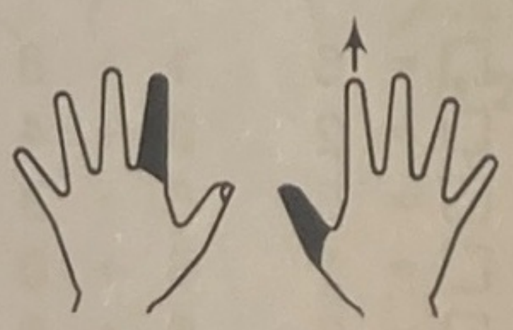
(f-3)



(f-4)

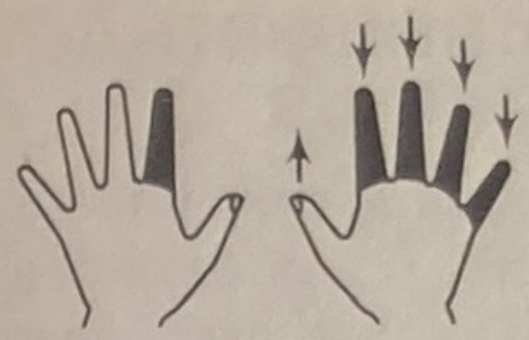
(g) Clear (one-by-one) from that number, (16), until you get a picture of the difference (9) on our fingers.

(h-1) Clear (saying): *"One"*



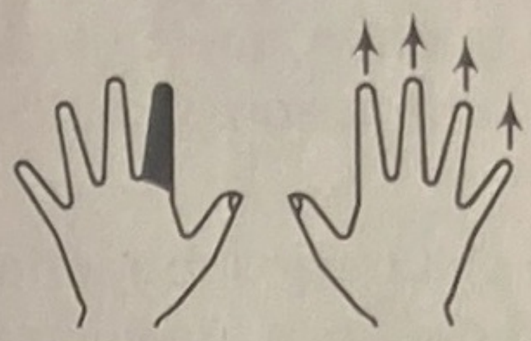
(h-1)

(h-2) Clear again (saying):
*"TWO"* (Through: Clear 5, Press 4)



(h-2)

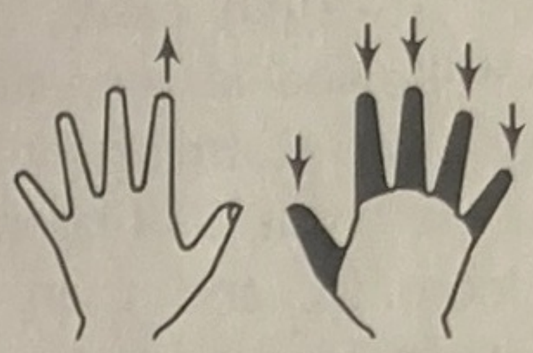
(h-3) Clear again (saying):
*"Three, Four, Five, Six"*



(h-3)

(h-4) Clear again (saying): *"SEVEN"*
(Through: Clear 10, Press 9)

After you say *"Seven,"* your fingers should look like illustration (a).



(h-4)

By counting to seven you have found your answer.

## PRACTICE EXERCISES

| | |
|---|---|
| 9 - □ + 6 = 8 | 1 - □ + 7 = 4 |
| 7 - □ + 8 = 9 | 4 - □ + 5 = 4 |
| 9 - □ + 7 = 8 | 0 - □ + 8 = 2 |
| 9 - □ + 6 = 6 | 2 - □ + 7 = 4 |
| 8 - □ + 6 = 8 | 1 - □ + 7 = 2 |
| 9 - □ + 5 = 7 | 1 - □ + 8 = 4 |
| 7 - □ + 7 = 9 | 1 - □ + 8 = 2 |
| 9 - □ + 6 = 7 | 2 - □ + 6 = 4 |
| 7 - □ + 5 = 5 | 1 - □ + 7 = 1 |
| 8 - □ + 8 = 9 | 6 - □ + 5 = 5 |
| 3 - □ + 5 = 7 | 4 - □ + 6 = 6 |
| 5 - □ + 4 = 6 | 9 - □ + 1 = 9 |

## MULTI-DIGIT SUBTRACTION—CALCULATIONS

One of the outstanding features of the CHISANBOP ™ instructional method for addition is that, even when the child is in Stage 1, he is able to handle examples with more than one column of figures. Therefore it should come as no surprise that, in subtraction, the child is also able to perform subtraction of multi-columned numbers.

Initially, in columnar subtraction, we consider only those cases in which the sum of the numbers to be subtracted from the minuend in any given column is always less than the number of the minuend in that column. In such cases, the procedure to be followed is very simple. The child does the subtraction, column by column. He reads his fingers as he finishes each column, and then writes the particular answer down at the bottom of the column. When the child has made all the required columnar subtractions, the answer to the problem is written down in front of him.

Make sure that you are calculating the following problems by columns. Up to now you have pressed the entire minuend. From here on you will be pressing only the units' digit of the minuend when you are calculating the units' column. Since no regrouping is required in the following problems, you will have no problems that require the Clear 10 Press 9 manipulation. The problems will have examples of the Clear 5 Press 4 manipulation.

When calculating the tens' column, you should initially count out the numbers by tens even though you are calculating on your right hand. Thus if the tens' digit of the minuend is a six, it should be pressed and counted as: "Onety, twenty, thirty, forty, fifty, sixty." If the tens' digit of the subtrahend is a three, it should be cleared and counted as; "Onety, twenty, thirty." Only after place value is firmly established can you go back to counting these numbers as if they were units.

The purpose of calculating these problems in a column format is to prepare the students for problems where regrouping is required and for problems where there are multi-digit minuends and subtrahends. Some students will be tempted to use the old method where they pressed the entire minuend. You must show care that they proceed according to the new format.

# CALCULATION BY COLUMN-BY-COLUMN WITHOUT REGROUPING

**Units' Column**

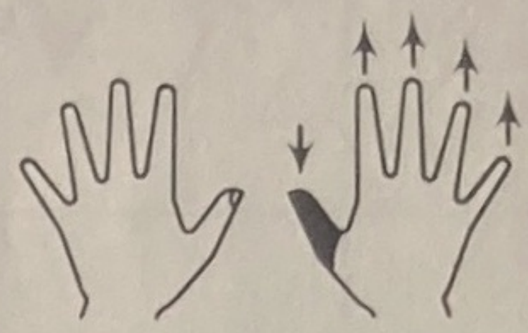
EXAMPLE:

| T | U |
|---|---|
| 6 | 8 |
| | −5 |
| □ | □ |

**(A) Press 8 one-by-one**

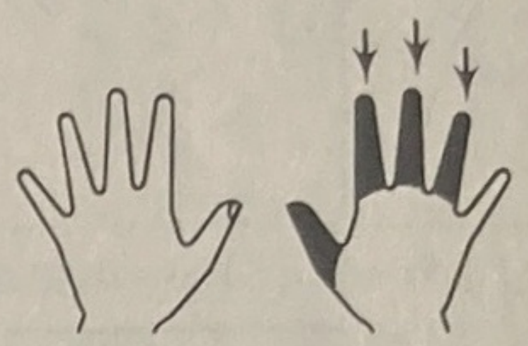
**A-1** (saying) *"One, Two, Three, Four, Five"*

**A-1**



**A-2** (saying) *"Six, Seven, Eight"*

**A-2**

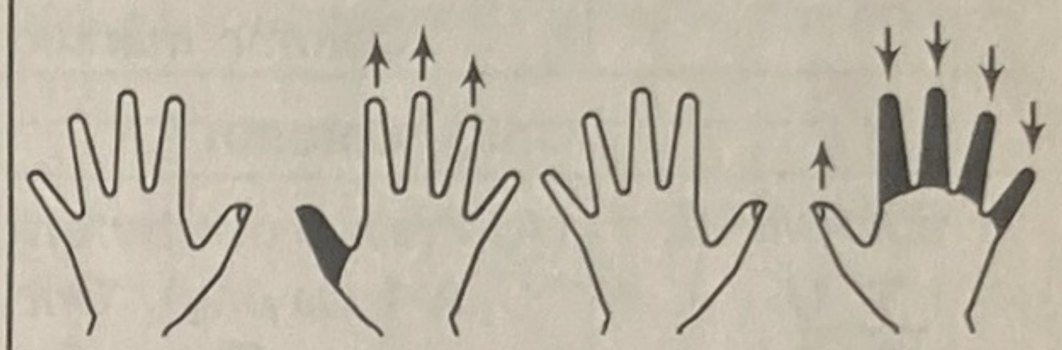


**(B) Clear 5 one-by-one**

**B-1** (saying) *"One, Two, Three"*

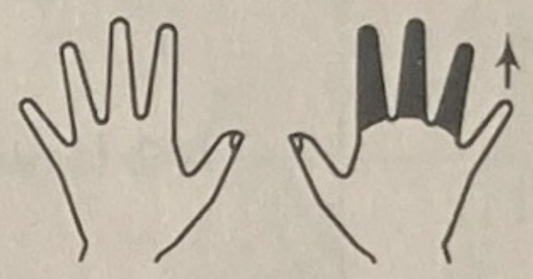
**B-2** (saying) *"Four"*

**B-1** **B-2**



**B-3** (saying) *"Five"*

**B-3**



This is the answer (3) for the Units' Column marked U—write 3 (ones) below the line in the units' place.

(Clear your fingers and proceed as follows with Tens' Column marked T)

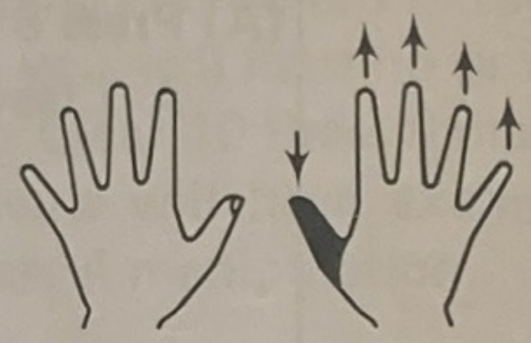
**Tens' Column**

T U
6 8
-5
□□

**(C) Press 6 ten-by-ten.**

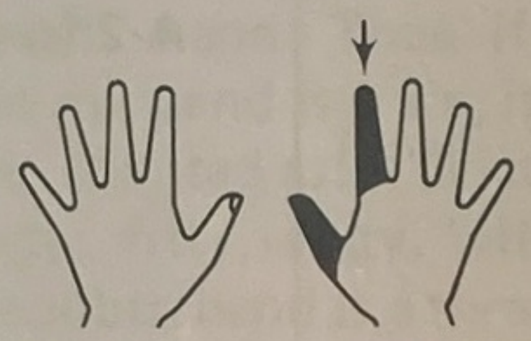
**C-1** (saying) *"Onety, Twenty, Thirty, Forty, Fifty"*

**C-1**



**C-2** (saying) *"Sixty"*

**C-2**



**C-3** Since there is no number in the Tens' Column on the bottom row (subtrahend), write what appears on your fingers, 6 (tens) in illustration C-2, which is the answer for the Tens' Column marked T.

**Units' Column**

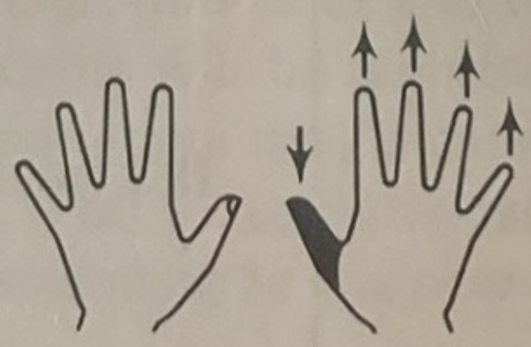
EXAMPLE:

T U
7 6
-4 3
□□

**(A) Press 6 one by one**

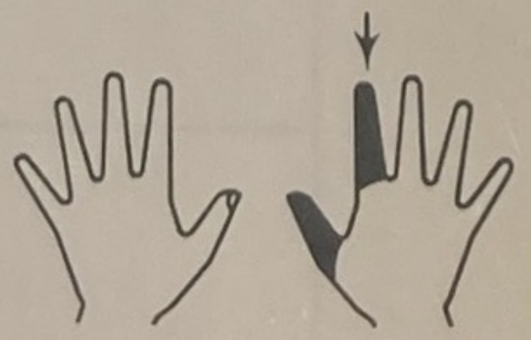
**A-1** (saying) *"One, Two, Three, Four, Five"*

**A-1**



**A-2** (saying) *"Six"*

**A-2**

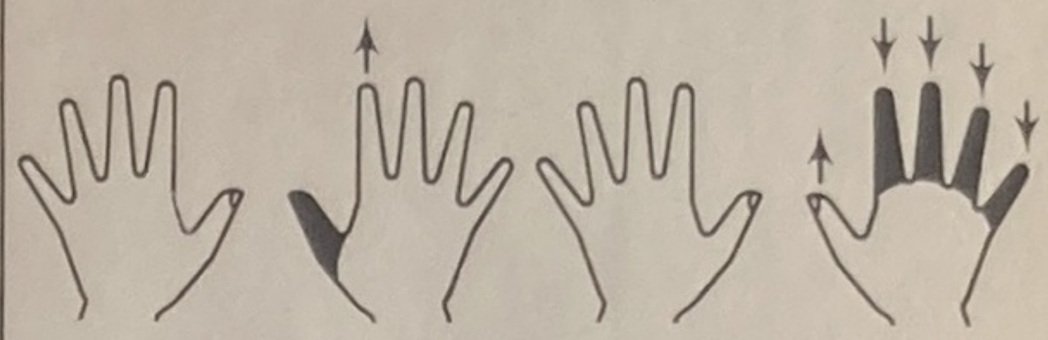


**(B) Clear 3 one-by-one**

**B-1** (saying) *"One"*

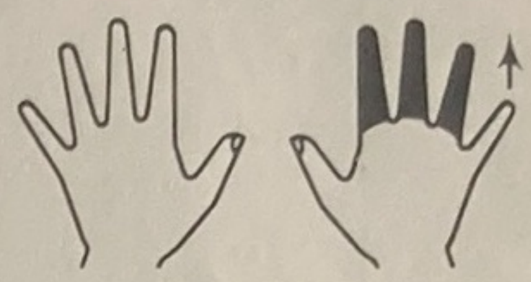
**B-2** (saying) *"Two"*

**B-1** **B-2**



**B-3** (saying) *"Three"*

**B-3**



This is the answer (3) for the Units' Column marked U. Write 3 (ones) below the line in the units' place.

(Clear your fingers and proceed as follows with Tens' Column marked T)

**Tens' Column**

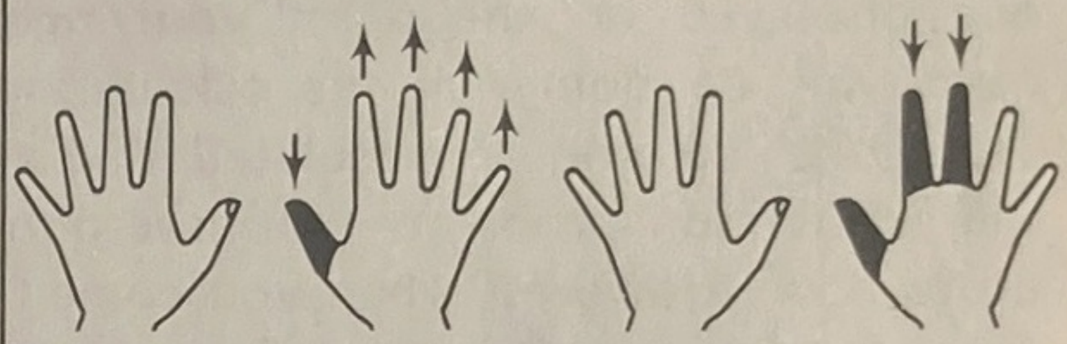
| T | U |
|---|---|
| 7 | 6 |
| -4 | 3 |
| □ | □ |

**(C) Press 7 ten-by-ten**

**C-1** (saying) *"Onety, Twenty, Thirty, Forty, Fifty"*

**C-2** (saying) *"Sixty, Seventy"*

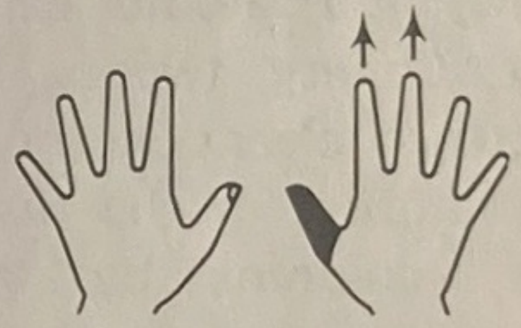
**C-1** **C-2**

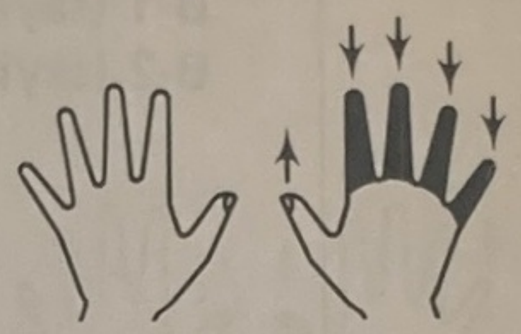
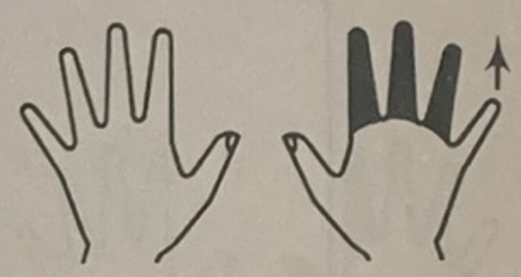


**(D) Clear 4 ten-by-ten**

**D-1** (saying) *"Onety, Twenty"*

**D-1**



| | |
|---|---|
| **D-2** (saying) *"Thirty"* | **D-2**  |
| **D-3** (saying) *"Forty"* | **D-3**  |
| | This is the answer, 3 (tens) for the Tens' Column marked T. Write 3 (tens) below the line in the tens' place. |

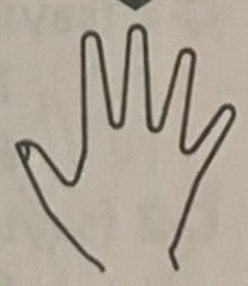
## CHANGING FINGER VALUES OF THE HANDS

### (Regrouping is not Required)

When you perform problems such as the one displayed on the right, your right hand will be doing all the calculating. The units' column is calculated like all the subtraction problems you have done up to now. However, when you come to the tens' column, the value of your hand changes. You press and clear the tens on your right hand. Since you are calculating tens, you should call them out verbally as "onety, twenty, etc" as you press and then clear. So too you should call the numbers in the hundreds' and thousands' columns by their proper names as you calculate those columns. Only after the students completely understand place value can you revert to the short cut of calling all the numbers by their unit value.

(A)

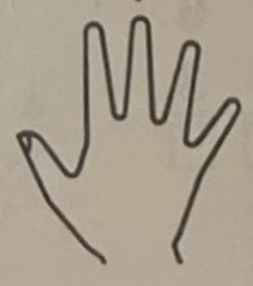
```
  9,482
- 5,231
```



(units)

(B)

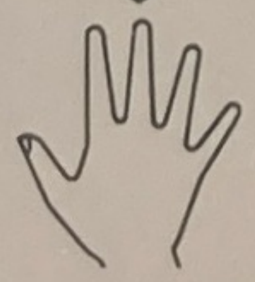
```
  9,482
- 5,231
```



(tens)

(C)

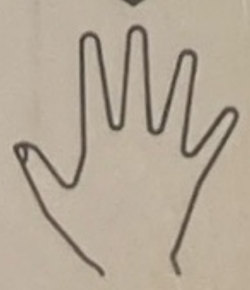
```
  9,482
- 5,231
```



(hundreds)

(D)

```
  9,482
- 5,231
```



(thousands)

## PRACTICE EXERCISES

Press the units' digit of the minuend and clear the units' digit of the subtrahend, one-by-one. Write the answer below the line and proceed to calculate the tens' column.

| | | | |
|---|---|---|---|
| 14 | 56 | 95 | 62 |
| - 3 | - 4 | - 3 | - 2 |
| 27 | 49 | 88 | 77 |
| - 6 | - 5 | - 7 | - 1 |
| 90 | 80 | 60 | 50 |
| -10 | -20 | -40 | -30 |
| 28 | 56 | 49 | 83 |
| -17 | -36 | -19 | -52 |
| 75 | 29 | 86 | 66 |
| -33 | -19 | -21 | -66 |

A 5 found in the tens' column of the minuend should be pressed and counted by tens as: "onety, twenty, thirty, forty, fifty." A 2 found in the tens' column of the subtrahend should be cleared and counted by tens as: "onety, twenty."

| | | | | |
|---|---|---|---|---|
| 94 | 15 | 18 | 27 | 80 |
| -83 | - 4 | - 5 | - 6 | -10 |
| 29 | 39 | 37 | 41 | 97 |
| -15 | - 6 | -32 | -21 | - 3 |
| 47 | 54 | 53 | 63 | 95 |
| -14 | - 4 | - 3 | - 3 | -50 |
| 63 | 72 | 76 | 84 | 11 |
| -53 | - 2 | -65 | -60 | -11 |

Be sure the students are calculating by columns and not reverting to the old format. When calculating the hundreds' column, use the proper counting, e.g. "one hundred, two hundred, etc."

| | | | |
|---|---|---|---|
| 22 | 36 | 75 | 84 |
| -11 | -24 | -43 | -62 |
| 48 | 51 | 97 | 38 |
| -16 | -30 | -75 | -28 |
| 913 | 865 | 736 | 528 |
| - 12 | - 51 | - 34 | - 25 |
| 156 | 267 | 481 | 509 |
| -123 | -154 | -271 | -308 |
| 117 | 256 | 709 | 668 |
| -103 | -245 | -607 | -161 |

Continue to calculate by columns. Press the digit of the minuend and clear the digit of the 1st subtrahend. Follow this by clearing the digit of the 2nd subtrahend.

| | | | |
|---|---|---|---|
| 98 | 99 | 86 | 97 |
| -11 | -20 | -20 | -41 |
| -12 | -23 | -10 | -21 |
| 88 | 44 | 84 | 66 |
| -12 | -12 | -30 | -32 |
| -21 | -11 | -31 | -21 |
| 66 | 77 | 77 | 85 |
| -12 | - 2 | -31 | - 2 |
| - 3 | -12 | - 4 | -33 |
| 88 | 88 | 99 | 57 |
| -52 | - 1 | -80 | - 5 |
| - 5 | -57 | - 9 | -32 |

# SUBTRACTION STAGE 1
# PART (B)

## Subtraction Requiring Regrouping

Most school children find the first lessons of arithmetic very easy to learn. They enjoy their work and take pride in their early success. But when they come to the topic of regrouping, many of them reach their first real stumbling block.

Unfortunately, some of them never master this key arithmetical operation. They are left with a feeling that mathematics is too hard to learn and give up trying to overcome this problem. Thus the first grader is often condemned to a lifetime of hating "math" and having little success with it.

Because the subject of regrouping is such a crucial matter in the development of Chisanbop (TM), we created a new method for teaching this difficult concept. The method is distinct from, and in addition to, the finger calculating techniques you have been learning.

Since the method differs to such an extent from the conventional method, we have chosen to call it the HYPAI TRADING.

The full value of the HYPAI TRADING will be understood only upon the completion of the discussion of regrouping. To facilitate the process, we have divided the material into two sections: HYPAI TRADING and CONSISTENT HYPAI TRADING.

## HYPAI TRADING

We will now show some typical subtraction problems and how they are handled under HYPAI TRADING. In subtraction, —and remember again we have consistently stressed that subtraction is the reverse of addition—we use a process called trading, the opposite of "carry-over" in addition. In addition, sometimes the student had to take (carry-over) *a ten* from the units' column into the tens' column where he placed *a one.* Here we reverse the process.

To simplify the trading process, we begin by expressing the subtraction problems on four lines instead of two.

The first and third lines are the TRADING lines. On the first line, we place square boxes for the *tens* [10] that are traded-in. On the third line, we place oblong boxes for the *ones* [-1] that are traded-out. On the second line we write the minuend. Finally the fourth line contains the subtrahend.

In the following problem, you must trade *one ten* [-1] from the tens' column for *a ten* [10] in the units' column. After you have performed the trade, you should proceed to find the answer for the units' column by adding *ten* [10] to the unit number of the minuend and subtracting the unit number in the subtrahend. You then proceed to perform the subtraction in the tens' column, making certain to take into account the [-1] in the trade-out box.

The First line [10]—Trade-in Boxes
The Second line 9 8—The Minuends
The Third line [-1] —Trade-out Boxes
The Fourth line - 7 9—The Subtrahends
1 9

This four line format only makes explicit what is implicit in conventional methods. Conventionally, students are taught to increase the units' column by ten and either

reduce the tens' digit of the minuend by one or increase the tens' digit of the subtrahend by one. See illustration.

**Chisanbop** ™

```
   10
  8 3
 -1
 -2 5
 ----
  5 8
```

**Conventional A**

```
  7 13
  8 3
 -2 5
 ----
  5 8
```

**Conventional B**

```
    13
  8 3
 -3
 -2 5
 ----
  5 8
```

Another feature of this rule is that it is simple and consistent. Each column must be examined to see if a trade is required. If students are unable to tell, they can always try without a trade using their fingers. When they run out of fingers before completion, they know a trade must be made. Moreover, the students are *always* proceeding from right to left. This is usually true of conventional approaches but *not always.* When subtracting 9 from 7000, conventionally, students must proceed from left to right. Young children find the consistent rule much more to their liking.

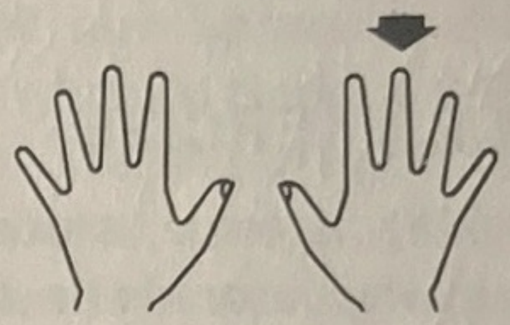
## CHANGING FINGER VALUES OF THE HAND

### (Regrouping is Required)

Since you are regrouping, both hands must be used in the calculating of each column. Be sure the students recognize that the value of the hands is changing. For instance when calculating the tens' column, the minuend should be pressed on the right hand, ten-by-ten, saying: "onety, twenty,...eighty." When you clear the trade-out number, it should be verbalized as: "onety" and the subtrahend should be cleared to the accompaniment of: "onety, twenty, thirty." Upon completion of the problem, ask the class, "How many tens on the right hand?" Ans. *"4 Tens."* "How many hundreds on the left hand?" Ans. *"No hundred* is on the left hand." The names for the numbers and the values of the hands change as you move from column to column. Only after you are absolutely convinced that the students fully understand place value can you use the unit names for the numbers in all the columns.

(A)

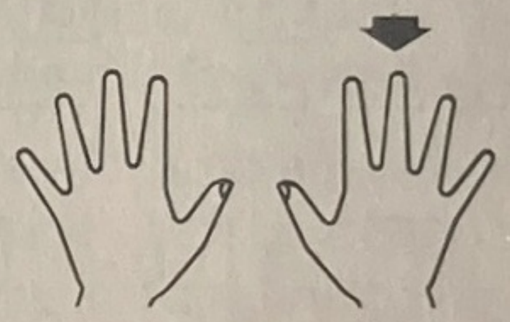
```
      10   10
  9, 4 8 2
 -1   -1
-  5, 7 3 6
 ---------
```



Tens Units

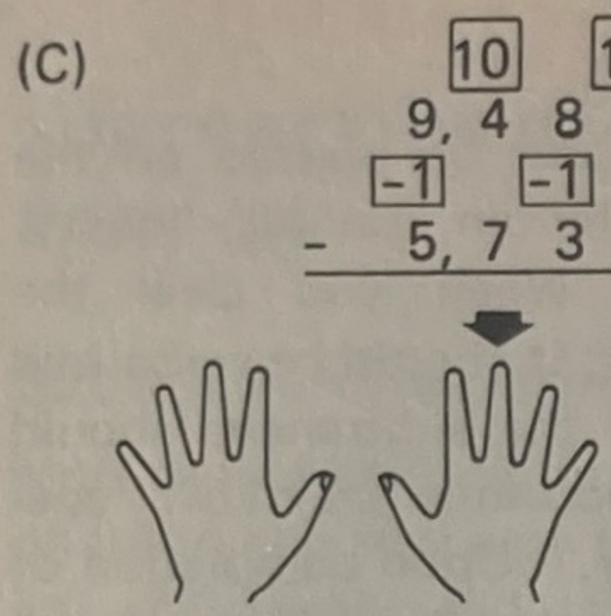
(B)

```
      10   10
  9, 4 8 2
 -1   -1
-  5, 7 3 6
 ---------
```

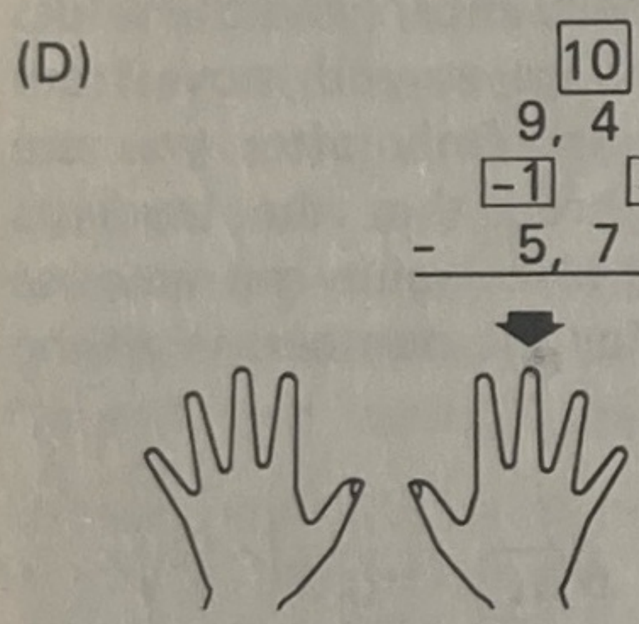


Hundreds Tens

(C)




(D)




## PRACTICE EXERCISES

Before making a trade, students should understand why a trade is required. If they can not see the reason, have them try to calculate the units' column on their fingers without making a trade. When they run out of fingers, they will understand.

| | | | | |
|---|---|---|---|---|
| 10 | 10 | 10 | 10 | 10 |
| 1 0 | 1 2 | 1 4 | 1 6 | 1 8 |
| -1 | -1 | -1 | -1 | -1 |
| - 3 | - 5 | - 7 | - 8 | - 9 |

| | | | | |
|---|---|---|---|---|
| □ | □ | □ | □ | □ |
| 2 1 | 2 3 | 2 5 | 2 7 | 2 8 |
| □ | □ | □ | □ | □ |
| - 3 | - 5 | - 7 | - 8 | - 9 |

| | | | | |
|---|---|---|---|---|
| □ | □ | □ | □ | □ |
| 3 0 | 3 2 | 3 4 | 3 6 | 3 8 |
| □ | □ | □ | □ | □ |
| - 2 | - 4 | - 6 | - 8 | - 9 |

| | | | | |
|---|---|---|---|---|
| □ | □ | □ | □ | □ |
| 4 1 | 4 2 | 4 3 | 4 4 | 4 5 |
| □ | □ | □ | □ | □ |
| - 5 | - 6 | - 7 | - 8 | - 9 |

| | | | | |
|---|---|---|---|---|
| □ | □ | □ | □ | □ |
| 5 0 | 5 2 | 5 4 | 5 6 | 5 8 |
| □ | □ | □ | □ | □ |
| - 2 | - 3 | - 7 | - 8 | - 9 |

The boxes have been printed for the convenience of the students. They should be aware that a trade is only made when it is required. These initial problems only have trades from the tens' column to the units' column, but the same technique is used later in other columns.

| | | | | |
|---|---|---|---|---|
| 10 | 10 | 10 | 10 | 10 |
| 6 1 | 6 3 | 6 5 | 6 7 | 6 8 |
| -1 | -1 | -1 | -1 | -1 |
| - 8 | - 9 | - 7 | - 8 | - 9 |

| | | | | |
|---|---|---|---|---|
| □ | □ | □ | □ | □ |
| 7 0 | 7 1 | 7 2 | 7 3 | 7 4 |
| □ | □ | □ | □ | □ |
| - 1 | - 2 | - 3 | - 4 | - 5 |

| | | | | |
|---|---|---|---|---|
| □ | □ | □ | □ | □ |
| 8 5 | 8 6 | 8 7 | 8 8 | 8 0 |
| □ | □ | □ | □ | □ |
| - 7 | - 8 | - 9 | - 9 | - 2 |

| | | | | |
|---|---|---|---|---|
| □ | □ | □ | □ | □ |
| 9 2 | 9 1 | 9 5 | 9 4 | 9 7 |
| □ | □ | □ | □ | □ |
| - 5 | - 4 | - 8 | - 7 | - 9 |

| | | | | |
|---|---|---|---|---|
| □ | □ | □ | □ | □ |
| 1 6 | 2 5 | 3 3 | 4 7 | 8 5 |
| □ | □ | □ | □ | □ |
| - 9 | - 8 | - 7 | - 8 | - 9 |

Compare the trade with the carry-over operation of addition. As in all other aspects, this subtraction method is the exact opposite of its counterpart in addition. It also can be compared to exchanging a dime for ten pennies, the value of the money remains the same.

| | | | | |
|---|---|---|---|---|
| 10 | 10 | 10 | 10 | 10 |
| 8 6 | 4 0 | 3 7 | 7 5 | 9 3 |
| -1 | -1 | -1 | -1 | -1 |
| -6 7 | -1 8 | -1 9 | -3 8 | -5 7 |
| □ | □ | □ | □ | □ |
| 7 2 | 5 1 | 6 2 | 3 2 | 2 7 |
| □ | □ | □ | □ | □ |
| -1 6 | -4 8 | -2 5 | -1 7 | -1 8 |
| □ | □ | □ | □ | □ |
| 6 5 | 9 4 | 3 6 | 2 8 | 5 5 |
| □ | □ | □ | □ | □ |
| -3 7 | -8 6 | -1 8 | -1 9 | -3 9 |
| □ | □ | □ | □ | □ |
| 2 7 | 8 6 | 3 1 | 6 7 | 7 4 |
| □ | □ | □ | □ | □ |
| -1 8 | -7 9 | -1 3 | -4 8 | -5 6 |
| □ | □ | □ | □ | □ |
| 5 0 | 6 0 | 7 0 | 8 0 | 9 0 |
| □ | □ | □ | □ | □ |
| -3 8 | -2 9 | -1 5 | -4 6 | -5 7 |

When no numbers appear in the hundreds' column of the subtrahend, you do not have to perform a calculation. You can bring the number from the minuend straight down below the line.

| | | | |
|---|---|---|---|
| 10 | 10 | 10 | 10 |
| 7 5 1 | 8 6 3 | 1 6 5 | 4 9 8 |
| -1 | -1 | -1 | -1 |
| -1 8 | -3 4 | -1 7 | -5 9 |
| □ | □ | □ | □ |
| 2 8 6 | 1 3 7 | 5 6 0 | 2 4 1 |
| □ | □ | □ | □ |
| -7 7 | -2 9 | -5 4 | -3 3 |

| | | | |
|---|---|---|---|
| □ | □ | □ | □ |
| 9 2 0 | 8 7 2 | 3 5 3 | 1 8 6 |
| □ | □ | □ | □ |
| -1 8 | -6 7 | -4 4 | -7 9 |
| □ | □ | □ | □ |
| 1 8 5 | 2 4 7 | 8 3 0 | 2 4 8 |
| □ | □ | □ | □ |
| -7 9 | -3 9 | -2 5 | -3 9 |

| | | | |
|---|---|---|---|
| □ | □ | □ | □ |
| 9 6 2 | 1 8 4 | 3 9 1 | 6 7 8 |
| □ | □ | □ | □ |
| -3 4 | -2 6 | -7 8 | -5 9 |

Up to now, you have only traded *a one* -1 from the tens' column for *a ten* 10 in the units' column. All the problems you have seen under regrouping have required this operation. However, you have not performed this operation for any of the other columns, nor have you faced situations where it might not be required. We now approach the question of deciding when to trade or regroup. For the moment, we will present this topic in the conventional manner. We do this so you may see the approach presented in traditional school textbooks. However once you master CONSISTENT HYPAI TRADING as shown in the next section, the question of whether or not to borrow is eliminated. The first part of this book contained examples that never required borrowing. The very last section of Subtraction Stage 1 (Part B) contained examples where it is always required.

Now we present problems where a choice must be made. We continue to present the problems in four lines instead of two. Thus the boxes in the top line once

again receive the tens [10] from the trade; the boxes in the third line contain the ones [-1] that are given away for the tens in the trade. The minuend is the second line and the subtrahend is the fourth line. This is where a new element is introduced.

Before proceeding to trade or regroup, you must first compare the numbers in the minuend with the numbers in the subtrahend, column-by-column. At each column, the student must ask himself, "Is the minuend bigger than the subtrahend?"

If the answer is yes, he proceeds to subtract immediately without performing the trading operation. If the answer is no, he must trade *a one* [-1] from the left hand column for *a ten* [10] in the column that he is examining.

If the students do not immediately recognize whether they need to trade, they can try the subtraction on their fingers. If they are able to subtract, they do not have to trade. If the students are unable to complete the subtraction, then they must trade.

## Column (A)

Let us examine the example on the right. The first column to look at is column (A). Comparing the numbers in the minuend and subtrahend, you now know that you cannot subtract 5 from 2. Thus you must make a HYPAI TRADE between columns (A) and (B). You place *a ten* [10] in the trade-in box in column (A) and [-1] in the trade-out box in column (B). You then complete the arithmetical operations to be performed in column (A) on your finger, arrive at the answer 7 and write that number below the line.

## Column (B)

(Note: there is a -4 from the subtrahend and [-1] from the trade-out box, both to be subtracted in column (B).)

When you look at column (B), you should recognize 6 is greater than 5 and proceed to subtract immediately. You write the answer 1 below the line.

## Column (C)

Upon comparing the minuend and subtrahends in column (C), you should recognize that once again you must perform a trade. Thus [10] is placed in the trade-in box in column (C) and [-1] is placed in the trade-out box in column (D). You then proceed to subtract in column (C) and get the answer 5 which you place below the line.

## Column (D)

Finally, you must compare the numbers in the minuend and subtrahend in column (D).

Since 5 is greater than 3 you subtract immediately and get the answer 2 which you write below the line (Note: There is a -2 from the subtrahend and [-1] from the trade-out box, both to be subtracted in column (D).)

EXAMPLE:

| (D) | (C) | (B) | (A) | |
|---|---|---|---|---|
| | ☐ | | ☐ | ← Trade-in |
| 5 | 3 | 6 | 2 | Boxes |
| ☐ | | ☐ | → | Trade-out |
| -2 | 8 | 4 | 5 | Boxes |

(B) (A)

10

6 2

-1

-4 5

7

HYPAI TRADE

(D) (C)

10

5 3

-1

-2 8

5

HYPAI TRADE

(B)

6

-1

-4

1

(D)

5

-1

-2

2

In the following pages, first examine each problem column-by-column. Ask yourself if you have to trade or regroup. After you learn to deal with this question correctly, you can proceed to complete the problems by performing the actual subtractions column-by-column.

Note that the problems on the last three pages do not contain the trade-in and trade-out boxes. They are not drawn in so that you make the trading decisions on your own.

Make sure you consistently make the right choice in deciding where to trade. In the first problems that follow, the boxes are provided for you as you begin to learn this process.

## PRACTICE EXERCISES

This is the first time you are trading from the hundreds' column into the tens' column. The students should realize that they are taking *one* hundred out of the hundreds' column and replacing it with *ten* tens. In terms of a consistent rule you always trade a -1 for a 10, but the implied meaning in terms of place value should also be explained.

| | | | |
|---|---|---|---|
| 10 | 10 | 10 | 10 |
| 6 3 5 | 7 0 3 | 1 0 8 | 9 1 6 |
| -1 | -1 | -1 | -1 |
| - 8 3 | - 9 2 | - 4 3 | - 5 4 |

| | | | |
|---|---|---|---|
| □ | □ | □ | □ |
| 2 2 2 | 5 7 6 | 9 2 4 | 8 4 9 |
| □ | □ | □ | □ |
| - 3 1 | - 9 5 | - 3 3 | - 6 8 |

| | | | |
|---|---|---|---|
| □ | □ | □ | □ |
| 3 7 5 | 6 7 8 | 5 2 1 | 3 1 2 |
| □ | □ | □ | □ |
| - 8 4 | - 9 7 | - 9 0 | - 7 2 |

| | | | |
|---|---|---|---|
| □ | □ | □ | □ |
| 1 1 1 | 7 2 4 | 5 0 0 | 6 0 1 |
| □ | □ | □ | □ |
| - 3 1 | - 8 2 | - 8 0 | - 9 1 |

| | | | |
|---|---|---|---|
| □ | □ | □ | □ |
| 6 7 5 | 3 4 9 | 1 0 6 | 7 5 3 |
| □ | □ | □ | □ |
| - 9 4 | - 5 7 | - 9 3 | - 8 0 |

This is the first time that two trades are required for a single problem. It gives you an excellent chance to review place value. When you are trading out of the tens into the units, you are trading *one* ten for *ten* units. When you are trading out of the hundreds into the tens you are trading *one* hundred or *ten* tens.

| | | | |
|---|---|---|---|
| 10 10 | 10 10 | 10 10 | 10 10 |
| 7 3 0 | 1 6 3 | 9 0 0 | 2 0 6 |
| -1 -1 | -1 -1 | -1 -1 | -1 -1 |
| - 8 9 | - 7 4 | - 8 9 | - 7 7 |

| | | | |
|---|---|---|---|
| □□<br>167<br>□□<br>−78 | □□<br>850<br>□□<br>−96 | □□<br>701<br>□□<br>−54 | □□<br>834<br>□□<br>−45 |
| □□<br>234<br>□□<br>−78 | □□<br>456<br>□□<br>−88 | □□<br>106<br>□□<br>−99 | □□<br>340<br>□□<br>−51 |
| □□<br>600<br>□□<br>−25 | □□<br>720<br>□□<br>−62 | □□<br>900<br>□□<br>−74 | □□<br>483<br>□□<br>−98 |
| □□<br>526<br>□□<br>−78 | □□<br>901<br>□□<br>−83 | □□<br>830<br>□□<br>−55 | □□<br>424<br>□□<br>−37 |

On first examination, only the units' column requires a trade. However, when you calculate the −1 traded out of the tens' column, that column also needs its own trade.

| | | | |
|---|---|---|---|
| 10□10□<br>725<br>−1□−1□<br>−27 | 10□10□<br>943<br>−1□−1□<br>−49 | 10□10□<br>180<br>−1□−1□<br>−89 | 10□10□<br>675<br>−1□−1□<br>−77 |
| □□<br>913<br>□□<br>−18 | □□<br>156<br>□□<br>−58 | □□<br>732<br>□□<br>−37 | □□<br>555<br>□□<br>−56 |
| □□<br>715<br>□□<br>−17 | □□<br>185<br>□□<br>−89 | □□<br>310<br>□□<br>−16 | □□<br>620<br>□□<br>−27 |
| □□<br>622<br>□□<br>−23 | □□<br>493<br>□□<br>−95 | □□<br>187<br>□□<br>−89 | □□<br>715<br>□□<br>−18 |

Whenever students are undecided whether or not a trade is required, have them attempt to calculate the column. If they run out of fingers before completing the calculation, a trade is needed.

| | | | |
|---|---|---|---|
| 10□10□<br>407<br>−1□−1□<br>−108 | 10□10□<br>806<br>−1□−1□<br>−508 | 10□10□<br>704<br>−1□−1□<br>−109 | 10□10□<br>206<br>−1□−1□<br>−107 |
| □□<br>923<br>□□<br>−124 | □□<br>740<br>□□<br>−346 | □□<br>810<br>□□<br>−516 | □□<br>408<br>□□<br>−109 |
| □□<br>297<br>□□<br>−198 | □□<br>382<br>□□<br>−186 | □□<br>256<br>□□<br>−158 | □□<br>364<br>□□<br>−167 |
| □□<br>811<br>□□<br>−718 | □□<br>723<br>□□<br>−628 | □□<br>256<br>□□<br>−157 | □□<br>206<br>□□<br>−109 |

The students are required to decide whether or not to trade. Be sure they are making correct decisions. If they aren't, it means that they do not fully understand what trading is all about and need further practice.

| | | | | |
|---|---|---|---|---|
| 10<br>32<br>−1<br>−18 | <br>56<br><br>−36 | 10<br>95<br>−1<br>−77 | <br>46<br><br>−23 | 10<br>68<br>−1<br>−49 |
| 65<br><br>−10 | 36<br><br>−13 | 97<br><br>−25 | 88<br><br>−13 | 77<br><br>−27 |

| | | | | |
|---|---|---|---|---|
| 8 8 | 5 1 | 3 2 | 2 7 | 7 5 |
| -1 9 | -1 0 | -1 3 | -1 6 | -3 9 |
| 7 9 | 4 7 | 9 2 | 5 5 | 2 1 |
| -3 9 | -2 8 | -6 2 | -3 6 | -1 7 |
| 6 9 | 7 5 | 9 7 | 5 9 | 4 7 |
| -3 6 | -8 1 | -1 3 | -1 9 | -2 5 |

Continue to check on whether the correct decisions are being made. If the students continue to have difficulty, have them try a column before trading. Only when they have to trade should a HYPAI Trade be made.

| | | | |
|---|---|---|---|
| 10 | 10 | 10 | 10 |
| 2 9 6 | 8 0 4 | 9 8 0 | 4 2 0 |
| -1 | -1 | -1 | -1 |
| - 3 7 | - 2 3 | - 1 6 | - 7 0 |
| 1 0 3 | 5 7 8 | 7 2 3 | 2 7 5 |
| - 2 2 | - 8 9 | - 9 0 | - 3 6 |
| 7 3 6 | 9 1 6 | 3 6 0 | 7 0 8 |
| -1 0 2 | -4 5 6 | -1 3 0 | -4 0 4 |
| 6 4 2 | 8 0 6 | 4 2 0 | 5 2 1 |
| - 6 9 | - 9 8 | - 7 3 | - 5 5 |
| 9 4 3 | 7 2 4 | 3 0 3 | 8 1 0 |
| -1 8 6 | -3 6 9 | -2 7 4 | -4 9 1 |

This section has the same digit appearing in both the minuend and the subtrahend of the tens' column. It is designed to show students how a trade in the units' column affects the tens' column.

| | | | |
|---|---|---|---|
| 10 10 | 10 10 | 10 10 | 10 10 |
| 9 0 4 | 9 5 8 | 6 3 0 | 5 4 5 |
| -1 -1 | -1 -1 | -1 -1 | -1 -1 |
| -7 0 5 | -3 5 9 | -2 3 7 | -3 4 8 |
| 6 9 3 | 5 7 6 | 4 1 7 | 6 3 1 |
| -2 9 4 | -1 7 9 | -2 1 8 | -2 3 6 |
| 7 0 0 | 8 1 6 | 5 4 3 | 8 6 7 |
| -4 0 6 | -4 1 8 | -2 4 5 | -5 6 8 |
| 2 1 8 | 3 4 5 | 4 5 2 | 3 0 0 |
| -1 1 9 | -2 4 7 | -1 5 6 | -1 0 1 |
| 7 0 4 | 4 8 6 | 5 0 7 | 2 2 2 |
| -5 0 6 | -1 8 9 | -4 0 8 | -1 2 4 |

Occasionally, the HYPAI Trade adds one or more implicit digits to the minuend. In the problems in this section, an additional trade has to be made because the tens' digit in the minuend is 0, and there is a -1 in that column from the 1st HYPAI Trade.

| | | | |
|---|---|---|---|
| 10 10 | □ □ | □ □ | □ □ |
| 1 0 0 | 4 0 3 | 6 0 7 | 8 0 1 |
| -1 -1 | □ □ | □ □ | □ □ |
| - 2 | - 8 | - 9 | - 4 |
| 10 10 | □ □ | □ □ | □ □ |
| 9 0 4 | 5 0 4 | 7 0 2 | 1 0 8 |
| -1 -1 | □ □ | □ □ | □ □ |
| - 6 | - 5 | - 7 | - 9 |

| | | | |
|---|---|---|---|
| 10 10<br>3 0 3<br>-1 -1<br>- 7 | □□<br>6 0 5<br>□□<br>- 8 | □□<br>1 0 4<br>□□<br>- 9 | □□<br>5 0 7<br>□□<br>- 9 |
| 10 10<br>7 0 1<br>-1 -1<br>- 6 | □□<br>4 0 2<br>□□<br>- 4 | □□<br>3 0 6<br>□□<br>- 8 | □□<br>9 0 0<br>□□<br>- 5 |

Trades have to be made into the tens' and hundreds' columns as a result of earlier trades. As long as each column is examined carefully, the decision of how many trades should be made will be clear.

| | | |
|---|---|---|
| 10 10 10<br>1, 0 0 0<br>-1 -1 -1<br>- 1 | □□□<br>2, 0 0 0<br>□□□<br>- 1 | □□□<br>3, 0 0 0<br>□□□<br>- 1 |
| 10 10 10<br>5, 0 0 0<br>-1 -1 -1<br>- 1 | □□□<br>6, 0 0 0<br>□□□<br>- 1 | □□□<br>7, 0 0 0<br>□□□<br>- 1 |
| 10 10 10<br>9, 0 0 0<br>-1 -1 -1<br>- 9 | □□□<br>8, 0 0 0<br>□□□<br>- 9 | □□□<br>7, 0 0 0<br>□□□<br>- 9 |
| 10 10 10<br>5, 0 0 0<br>-1 -1 -1<br>- 9 | □□□<br>4, 0 0 0<br>□□□<br>- 9 | □□□<br>3, 0 0 0<br>□□□<br>- 9 |
| □□□<br>4, 0 0 0<br>□□□<br>- 1 | □□□<br>6, 0 0 0<br>□□□<br>- 9 | |
| □□□<br>8, 0 0 0<br>□□□<br>- 1 | □□□<br>2, 0 0 0<br>□□□<br>- 9 | |

HYPAI Trades are not made in every column. Usually it is only for each digit in the subtrahends. Occasionally, there are more as you account for the –1's in the trade-out boxes. Once all the –1's are accounted for, you can discontinue trading.

□□□
1, 0 0 1, 0 0 0
□□□
- 1

□□□
3, 0 0 3, 0 0 0
□□□
- 1

□□□
4, 0 0 4, 0 0 0
□□□
- 1

□□□
6, 0 0 6, 0 0 0
□□□
- 1

□□□
2, 0 0 2, 0 0 0
□□□
- 1

□□□
3, 0 0 3, 0 0 0
□□□
- 5

□□□
5, 0 0 5, 0 0 0
□□□
- 1

□□□
6, 0 0 6, 0 0 0
□□□
- 5

```
  □□□
7,007,000
   □□□
-      1
```

```
  □□□
9,009,000
   □□□
-      1
```

```
  □□□
9,001,000
   □□□
-      9
```

```
  □□□
7,007,000
   □□□
-      9
```

```
  □□□
8,008,000
   □□□
-      1
```

```
  □□□
9,009,000
   □□□
-      5
```

```
  □□□
8,008,000
   □□□
-      9
```

```
  □□□
7,007,000
   □□□
-      5
```

This provides more practice with problems that call for additional trades. The general rule with this kind of problem is to calculate, column-by-column. When you find that an additional trade is called for, make the additional trade.

```
 10 10 10 10
 2 0,0 0 3
-1 -1 -1 -1
-      1 6
```

```
 10 10 10 10
 9 0,0 0 0
-1 -1 -1 -1
-      1 1
```

```
 10 10 10 10
 3 0,0 1 4
-1 -1 -1 -1
-      5 9
```

```
 10 10 10 10
 1 0,0 9 1
-1 -1 -1 -1
-      9 2
```

```
  □□□□
 7 0,0 0 5
  □□□□
-      9 7
```

```
  □□□□
 5 0,0 3 2
  □□□□
-      6 7
```

```
  □□□□
 4 0,0 0 6
  □□□□
-      5 8
```

```
  □□□□
 8 0,0 2 7
  □□□□
-      9 8
```

```
  □□□□
 1 0,0 3 2
  □□□□
-      4 8
```

```
  □□□□
 4 0,0 7 6
  □□□□
-      8 9
```

```
  □□□□
 6 0,0 3 4
  □□□□
-      7 5
```

```
  □□□□
 6 0,0 7 0
  □□□□
-      9 1
```

## CONSISTENT HYPAI TRADING

We now consider the CONSISTENT HYPAI TRADING. We are sure you will find its consistency and simplicity remarkable. There is only one rule to remember: "Always trade." The overriding virtue of the HYPAI TRADING concept is that it offers a consistent and regular approach to all subtraction problems. No longer does the child have to examine the individual columns of the minuend and the subtrahend to determine whether he must regroup or not. Instead, under the CONSISTENT HYPAI TRADING approach to subtraction, all problems are handled uniformly.

You will find that younger children are more comfortable with this consistent approach than with other methods. The method works even with slow learners who may never understand what occurs when regrouping is required, but are able to get the right answer once a simple and *consistent* approach is given to them. Experience has proven that much of the fear shown by children when they are first taught subtraction in the conventional method disappears once they are introduced to the CONSISTENT HYPAI TRADING.

Before we look at specific examples, we must refine slightly our four line approach to subtraction. Since we will trade whether or not the number in a minuend column is less than the number in the corresponding subtrahend column, we must allow for the fact that some column answers will yield two-digit numbers. Thus, at the top of the problem, we add a fifth line for carry-overs. Paradoxically, the additional row does not add to our work but simplifies it. The consistent approach makes the quest for an answer very simple.

In the example below, the first step is to make the consistent trades for each digit in the subtrahend. Thus a ten [10] should be placed in the units' column and a [-1] should be placed in the tens' column. This should be followed by placing a ten [10] in the tens' column and a [-1] in the hundreds' column. Again, a ten [10] should be placed in the hundreds' column and a [-1] in the thousands' column. Finally a ten [10] should be placed in the thousands' column and a [-1] in the ten thousands' column.

CONSISTENT HYPAI TRADES

| | (E) | (D) | (C) | (B) | (A) |
|---|---|---|---|---|---|
| Carry-over ← | □ | □ | □ | □ | |
| Trade-in ← | | □ | □ | □ | □ |
| Minuends ← | | 6 | 0 | 8 | 7 |
| Trade-out ← | □ | □ | □ | □ | |
| Subtrahends ← | - | 3 | 7 | 5 | 6 |

When they make the trades, some students will be tempted to first write in all the 10's and then to write in all the -1's. However, it is highly recommended that the numbers be written in the order described. As you are writing in the numbers, you should verbalize what is happening numerically. Thus you should be saying something like: "First we trade a ten for ten units; we follow this by trading a hundred for ten tens; then, we trade a thousand for ten hundreds, finally we trade a ten thousand for ten thousands."

If you have made the trades properly, you should have placed 10's in the trade-in boxes of Columns (A), (B), (C), and (D) and -1's in the trade-out boxes of Columns (B), (C), (D), and (E). If your problem looks like the one below, you can now proceed with your calculations.

| (E) | (D) | (C) | (B) | (A) |
|---|---|---|---|---|
| □ | □ | □ | □ | |
| | [10] | [10] | [10] | [10] |
| | 6 | 0 | 8 | 7 |
| [-1] | [-1] | [-1] | [-1] | |
| - | 3 | 7 | 5 | 6 |

## For Column (A)

First press the [10] from the trade-in box;

press another 7 from the minuend. Finally clear the 6 shown in the subtrahend. Read your fingers.

Since the answer is 11, write 1 below the line. Then write a 1 for the ten registered on the left hand into the carry-over box in Column B.

## For Column (B)

First press a 1 from the carry-over box and then the [10] from the trade-in. Press the 8 from the minuend; clear the [-1] from the trade-out box and clear another 5 from the subtrahend. Since your fingers now read 13, a 3 should be placed below the line and a 1 should be placed into the carry-over box in Column (C).

## For Column (C)

First press the 1 from the carry-over and then the [10] from the trade-in. Since the digit in the minuend is 0, you do not press for that entry. You now must clear the [-1] from the trade-out box and another 7 found in the subtrahend. The answer 3 is written below the line. Since there is nothing registered on the left hand, a 0 should be placed into the carry-over box in Column (D).

## For Column (D)

First press the [10] from the trade-in box and then the 6 of the minuend. Clear the [-1] found in the trade-out box and the 3 from the subtrahend. Your fingers should now read 12, so a 2 should be placed below the line and a 1 should go into the carry-over box in Column (E).

## For Column (E)

The 1 in the carry-over box cancels with the [-1] in the trade-out box and no entry has to be made below the line. Having completed the problem, your work should look like the example below.

| (E) | (D) | (C) | (B) | (A) |
|---|---|---|---|---|
| [1] | [0] | [1] | [1] | |
| | [10] | [10] | [10] | [10] |
| | 6 | 0 | 8 | 7 |
| [-1] | [-1] | [-1] | [-1] | |
| - | 3 | 7 | 5 | 6 |
| | 2 | 3 | 3 | 1 |

## PRACTICE EXERCISES

A trade must be made into any column where there is a digit in the subtrahend. Students have no difficulty with -1 in the tens' column if they calculate, column-by-column.

| | | | | |
|---|---|---|---|---|
| [ ] | [ ] | [ ] | [ ] | [ ] |
| [10] | [ ] | [ ] | [ ] | [ ] |
| 6 | 5 | 9 | 7 | 8 |
| [-1] | [ ] | [ ] | [ ] | [ ] |
| - 1 | - 2 | - 3 | - 1 | - 6 |
| [ ] | [ ] | [ ] | [ ] | [ ] |
| [10] | [ ] | [ ] | [ ] | [ ] |
| 8 | 5 | 9 | 9 | 9 |
| [-1] | [ ] | [ ] | [ ] | [ ] |
| - 4 | - 3 | - 2 | - 4 | - 2 |
| [ ] | [ ] | [ ] | [ ] | [ ] |
| [10] | [ ] | [ ] | [ ] | [ ] |
| 6 | 8 | 3 | 2 | 6 |
| [-1] | [ ] | [ ] | [ ] | [ ] |
| - 1 | - 5 | - 2 | - 1 | - 5 |
| [ ] | [ ] | [ ] | [ ] | [ ] |
| [10] | [ ] | [ ] | [ ] | [ ] |
| 4 | 9 | 8 | 7 | 5 |
| [-1] | [ ] | [ ] | [ ] | [ ] |
| - 3 | - 6 | - 6 | - 3 | - 1 |

When the subtrahend is a single-digit number, a 10 must be traded out of the tens' column into the units' column. Make this trade even in places where regrouping normally would not be required.

| | | | | |
|---|---|---|---|---|
| □ | □ | □ | □ | □ |
| 10 | □ | □ | □ | □ |
| 6 0 | 5 7 | 7 9 | 6 5 | 8 2 |
| -1 | □ | □ | □ | □ |
| - 4 | - 9 | - 8 | - 7 | - 3 |
| □ | □ | □ | □ | □ |
| 10 | □ | □ | □ | □ |
| 7 3 | 9 7 | 8 7 | 3 0 | 7 3 |
| -1 | □ | □ | □ | □ |
| - 8 | - 9 | - 7 | - 1 | - 5 |
| □ | □ | □ | □ | □ |
| 10 | □ | □ | □ | □ |
| 8 6 | 1 7 | 3 2 | 4 5 | 6 8 |
| -1 | □ | □ | □ | □ |
| - 5 | - 6 | - 7 | - 1 | - 4 |
| □ | □ | □ | □ | □ |
| 10 | □ | □ | □ | □ |
| 7 5 | 6 9 | 7 0 | 3 6 | 2 9 |
| -1 | □ | □ | □ | □ |
| - 3 | - 8 | - 1 | - 4 | - 3 |

When the subtrahend has two digits, make a trade into both the units' and the tens' columns. This consistent rule eliminates the requirement of having to decide whether or not to regroup.

| | | |
|---|---|---|
| □□ | □□ | □□ |
| 10 10 | □□ | □□ |
| 7 8 | 8 3 | 5 6 |
| -1 -1 | □□ | □□ |
| - 6 9 | - 2 7 | - 4 0 |
| □□ | □□ | □□ |
| 10 10 | □□ | □□ |
| 9 0 | 7 2 | 7 3 |
| -1 -1 | □□ | □□ |
| - 4 8 | - 1 3 | - 4 1 |

| | | |
|---|---|---|
| □□ | □□ | □□ |
| 10 10 | □□ | □□ |
| 7 8 | 9 0 | 3 7 |
| -1 -1 | □□ | □□ |
| - 4 6 | - 1 0 | - 2 8 |
| □□ | □□ | □□ |
| 10 10 | □□ | □□ |
| 7 7 | 2 6 | 6 7 |
| -1 -1 | □□ | □□ |
| - 4 3 | - 1 0 | - 3 9 |
| □□ | □□ | □□ |
| 10 10 | □□ | □□ |
| 9 0 | 5 8 | 8 9 |
| □□ | □□ | □□ |
| - 3 5 | - 1 9 | - 3 1 |

Whenever making a trade, verbalize the value that is being moved from column to column, e.g. "*One* ten is traded for ten units; *one* hundred is traded for ten tens."

| | | |
|---|---|---|
| □□ | □□ | □□ |
| 10 10 | □□ | □□ |
| 7 3 2 | 6 0 7 | 7 1 3 |
| -1 -1 | □□ | □□ |
| - 9 8 | - 8 9 | - 5 4 |
| □□ | □□ | □□ |
| 10 10 | □□ | □□ |
| 9 1 8 | 6 3 5 | 9 1 0 |
| -1 -1 | □□ | □□ |
| - 6 2 | - 7 4 | - 4 0 |
| □□ | □□ | □□ |
| 10 10 | □□ | □□ |
| 6 7 5 | 7 6 2 | 8 5 7 |
| -1 -1 | □□ | □□ |
| - 3 8 | - 4 9 | - 3 8 |
| □□ | □□ | □□ |
| 10 10 | □□ | □□ |
| 9 5 2 | 8 7 3 | 1 6 3 |
| -1 -1 | □□ | □□ |
| - 8 7 | - 1 1 | - 5 3 |

$$\begin{array}{r} \square\square \\ \square\square \\ \hline 510 \\ -\ \square\square \\ \hline -\ 82 \end{array} \qquad \begin{array}{r} \square\square \\ \square\square \\ \hline 567 \\ -\ \square\square \\ \hline -\ 95 \end{array} \qquad \begin{array}{r} \square\square \\ \square\square \\ \hline 724 \\ -\ \square\square \\ \hline -\ 16 \end{array}$$

$$\begin{array}{r} \square \\ \hline 35 \\ -\ \square \\ \hline -\ 6 \end{array} \qquad \begin{array}{r} \square \\ \hline 34 \\ -\ \square \\ \hline -\ 9 \end{array} \qquad \begin{array}{r} \square\square \\ \hline 10\,10 \\ 85 \\ -1\,-1 \\ \hline -\ 13 \end{array}$$

When the subtrahend has three digits, then three trades must be made. One easy consistent rule becomes an aid for students who require remedial attention.

$$\begin{array}{r} \square\square \\ \hline 96 \\ -\ 46 \\ \hline \square\square \\ \hline 64 \\ -\ 31 \\ \hline \square\square\square \\ \hline 736 \\ -\ 47 \\ \hline \square\square\square \\ \hline 296 \\ -\ 129 \end{array} \qquad \begin{array}{r} \square\square \\ \hline 29 \\ -\ 17 \\ \hline \square\square \\ \hline 10\,10 \\ 704 \\ -1\,-1 \\ \hline -\ 69 \\ \hline \square\square\square \\ \hline 324 \\ -\ 35 \\ \hline \square\square\square \\ \hline 306 \\ -\ 173 \end{array} \qquad \begin{array}{r} \square\square \\ \hline 85 \\ -\ 44 \\ \hline \square\square \\ \hline 418 \\ -\ 59 \\ \hline \square\square\square \\ \hline 10\,10\,10 \\ 794 \\ -1\,-1\,-1 \\ \hline -\ 185 \\ \hline \square\square\square \\ \hline 827 \\ -\ 436 \end{array}$$

$$\begin{array}{r} \square\square\square \\ \hline 10\,10\,10 \\ 521 \\ -1\,-1\,-1 \\ \hline -\ 310 \\ \hline \square\square\square \\ \hline 10\,10\,10 \\ 431 \\ -1\,-1\,-1 \\ \hline -\ 185 \\ \hline \square\square\square \\ \hline 10\,10\,10 \\ 416 \\ -1\,-1\,-1 \\ \hline -\ 153 \\ \hline \square\square\square \\ \hline 10\,10\,10 \\ 569 \\ -1\,-1\,-1 \\ \hline -\ 325 \\ \hline \square\square\square \\ \hline 333 \\ -\ \square\square\square \\ \hline -\ 124 \end{array} \qquad \begin{array}{r} \square\square\square \\ \hline 804 \\ -\ \square\square\square \\ \hline -\ 187 \\ \hline \square\square\square \\ \hline 693 \\ -\ \square\square\square \\ \hline -\ 157 \\ \hline \square\square\square \\ \hline 538 \\ -\ \square\square\square \\ \hline -\ 253 \\ \hline \square\square\square \\ \hline 689 \\ -\ \square\square\square \\ \hline -\ 164 \\ \hline \square\square\square \\ \hline 714 \\ -\ \square\square\square \\ \hline -\ 386 \end{array} \qquad \begin{array}{r} \square\square\square \\ \hline 525 \\ -\ \square\square\square \\ \hline -\ 363 \\ \hline \square\square\square \\ \hline 390 \\ -\ \square\square\square \\ \hline -\ 189 \\ \hline \square\square\square \\ \hline 905 \\ -\ \square\square\square \\ \hline -\ 183 \\ \hline \square\square\square \\ \hline 387 \\ -\ \square\square\square \\ \hline -\ 136 \\ \hline \square\square\square \\ \hline 614 \\ -\ \square\square\square \\ \hline -\ 393 \end{array}$$

Compare *trading* in subtraction to *carrying-over* in addition. When you have carry-over numbers, you remove one or more tens from the units' column and place them into the tens' column. When you trade, you remove a ten from the tens' column and place it in the units' column.

$$\begin{array}{r} \square \\ \hline 10 \\ 30 \\ -1 \\ \hline -\ 1 \\ \hline \square \\ \hline 42 \\ -\ 6 \\ \hline \square\square \\ \hline 10\,10 \\ 72 \\ -1\,-1 \\ \hline -\ 51 \end{array} \qquad \begin{array}{r} \square \\ \hline 23 \\ -\ 5 \\ \hline \square \\ \hline 39 \\ -\ 9 \end{array} \qquad \begin{array}{r} \square \\ \hline 58 \\ -\ 7 \end{array}$$

The trade-in and trade-out boxes have been intentionally eliminated so you can practice deciding where the trade numbers should be placed.

$$\begin{array}{r} \square \\ \hline 10 \\ 73 \\ -1 \\ \hline -\ 7 \end{array} \qquad \begin{array}{r} \square \\ \hline 90 \\ -\ 8 \end{array} \qquad \begin{array}{r} \square \\ \hline 62 \\ -\ 7 \end{array}$$

| | | |
|---|---|---|
| □□ | □□ | □□ |
| 67 | 22 | 40 |
| - 33 | - 15 | - 30 |
| □□ | □□ | □□ |
| | 10 10 | |
| 69 | 609 | 635 |
| | -1 -1 | |
| - 41 | - 83 | - 75 |
| □□ | □□ | □□□ |
| | | 10 10 10 |
| 947 | 888 | 593 |
| | | -1 -1 -1 |
| - 57 | - 95 | - 185 |
| □□□ | □□□ | □□□ |
| 439 | 356 | 744 |
| - 187 | - 137 | - 580 |

## EXPANDED TRADING (EXPANDED SUBTRAHEND)

The consistent trading rule, up to now, has been to make a trade for every digit of the subtrahend. In a very short time, we will be changing the format of the trading so as to enable you to calculate problems with 2 or more subtrahends or to calculate problems where the subtrahend is stated first. There, you will be trading based on the digits of the minuend.

3,096,798
- 7,342

Yet, in a problem such as the one in the above illustration, the speed of calculation relies on the fact that you only perform 4 sets of trades (i.e. trading according to digits in the subtrahend).

However, there is one particular set of problems, *easily recognizable,* which demands additional trades. Until you reach the new format in the later pages, you must make a few adjustments. Consider the problem in the illustration below.

| | |
|---|---|
| | 0 1 |
| | 10 10 |
| 2058 | |
| | -1 -1 |
| - | 75 |
| | 83 |

After you have made the usual trades and computed the units' and tens' columns, you are left with a -1 in the hundreds' column and no positive numbers in that column to compensate for the trade-out number. The following pages discuss what to do in such a situation.

There are several approaches to handle problems of this nature. In the illustrated problem, one method would be to start calculating, as usual, using the Consistent HYPAI Trades. It is only when you reach the hundreds' column that you must make an adjustment. Examining the hundreds' column, you can treat the -1 from the trade-out box as an expanding subtrahend, i.e. an implicit number in the subtrahend which demands its own HYPAI Trade. Thus, although the initial impression upon examining the problem was that trades were required only for the units' and tens' column, a third trade is needed for the hundreds' column.

A second method of treating this problem is to have the students cover the tens' and units' columns with their hands or paper after they have computed those columns. The unsolved part of the problem now becomes 20 - 1, a problem

they were able to solve back in the very early stages of their introduction to Chisanbop (TM) Subtraction. They probably have that number fact internalized and if not, can easily establish it.

The examples below show how various expanded trades are required for various problems.

## No extra trades because of carry-over number

No Extra Trades ←

```
    [1]
   [10]
 4, 7 0 8
    [-1]
-          5
```

No Extra Trades ←

```
   [1][0]
  [10][10]
 4, 0 7 5
   [-1][-1]
-        3 9
```

## One extra trade required

One Extra Trade ←

```
   [0][0]
  [10][10]
 4, 7 0 | 5
   [-1][-1]
-         | 8
```

One Extra Trade ←

```
   [0][0][0]
     [10][10][10]
 6, 0 0 2, 0 0 | 5
        [-1][-1][-1]
-               | 2 7
```

## Two extra trades required

Two Extra Trades ←

```
   [0][0][0]
     [10][10][10]
 4, 0 0 | 3
   [-1][-1][-1]
-        | 6
```

Two Extra Trades ←

```
   [0][0][0]
     [10][10][10]
 5, 0 0 4, 0 0 | 3
        [-1][-1][-1]
-               | 9
```

## PRACTICE EXERCISES

All these problems require an additional trade in the tens' column. As long as you concentrate on calculating column-by-column, this added operation should not prove very hard to grasp.

```
   [0][0]
  [10][10]
 2, 4 0 | 6
   [-1][-1]
-         | 9
```

```
   [ ][ ]
  [ ][ ]
 8, 9 0 | 5
   [ ][ ]
-         | 7
```

```
   [0][0]
  [10][10]
 5, 1 0 0
   [-1][-1]
-          4
```

```
   [ ][ ]
  [ ][ ]
 6, 2 0 | 4
   [ ][ ]
-         | 8
```

```
   [0][0]
  [10][10]
 7, 6 0 | 2
   [-1][-1]
-         | 3
```

```
   [ ][ ]
  [ ][ ]
 5, 6 0 | 1
   [ ][ ]
-         | 2
```

```
   [0][0]
  [10][10]
 3, 4 0 | 6
   [-1][-1]
-         | 7
```

```
   [ ][ ]
  [ ][ ]
 9, 1 0 | 2
   [ ][ ]
-         | 4
```

□□
□□
3, 7 0 0
–  6

□□
□□
9, 2 0 1
–  5

□□
□□
8, 5 0 0
–  1

□□
□□
1, 2 0 5
–  9

Two extra trades are required in the following problems. If your students are able, you can have them write the answer to 200 – 1 directly instead of performing the extra trades in the first problem.

0 0 0
10 10 10
2, 0 0 0
–1 –1 –1
–  9

□□□
□□□
8, 0 0 0
□□□
–  7

0 0 0
10 10 10
5, 0 0 0
–1 –1 –1
–  4

□□□
□□□
6, 0 0 0
□□□
–  8

0 0 0
10 10 10
7, 0 0 0
–1 –1 –1
–  3

□□□
□□□
5, 0 0 0
□□□
–  2

0 0 0
10 10 10
3, 0 0 0
–1 –1 –1
–  7

□□□
□□□
9, 0 0 0
□□□
–  4

□□□
□□□
3, 0 0 0
□□□
–  6

□□□
□□□
9, 0 0 0
□□□
–  5

□□□
□□□
8, 0 0 0
□□□
–  1

□□□
□□□
1, 0 0 0
□□□
–  9

One extra trade is required in these problems, but the students should be able to perform the calculation without making the trade.

0 0 1
10 10 10
1, 0 0 2, 0 0 8
–1 –1 –1
–  6 4

0 0 1
10 10 10
3, 0 0 3, 0 0 7
–1 –1 –1
–  1 6

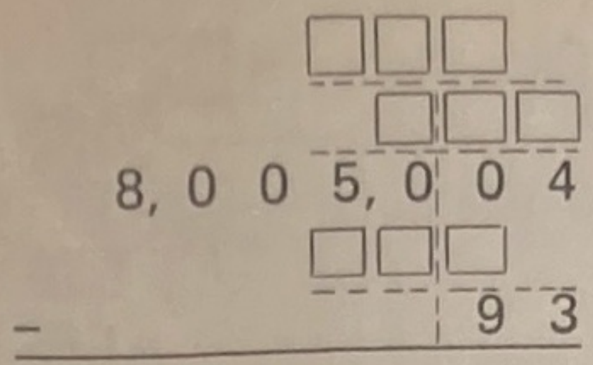
0 0 1
10 10 10
5, 0 0 6, 0 0 9
–1 –1 –1
–  3 5

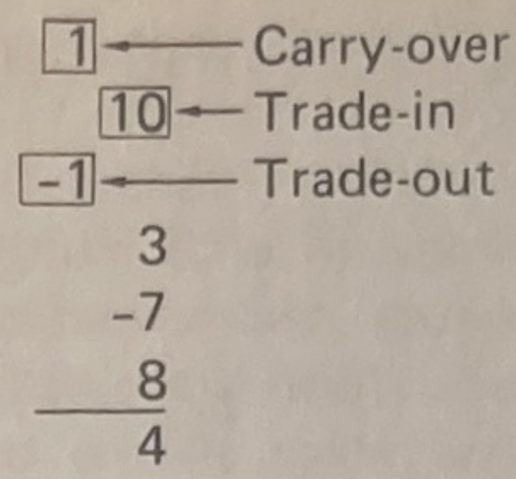
0 0 1
10 10 10
7, 0 0 7, 0 0 5
–1 –1 –1
–  4 2

□□□
□□□
2, 0 0 8, 0 0 9
□□□
–  2 7

□□□
□□□
4, 0 0 1, 0 0 6
□□□
–  8 4

□□□
□□□
6, 0 0 4, 0 0 3
□□□
–  5 1



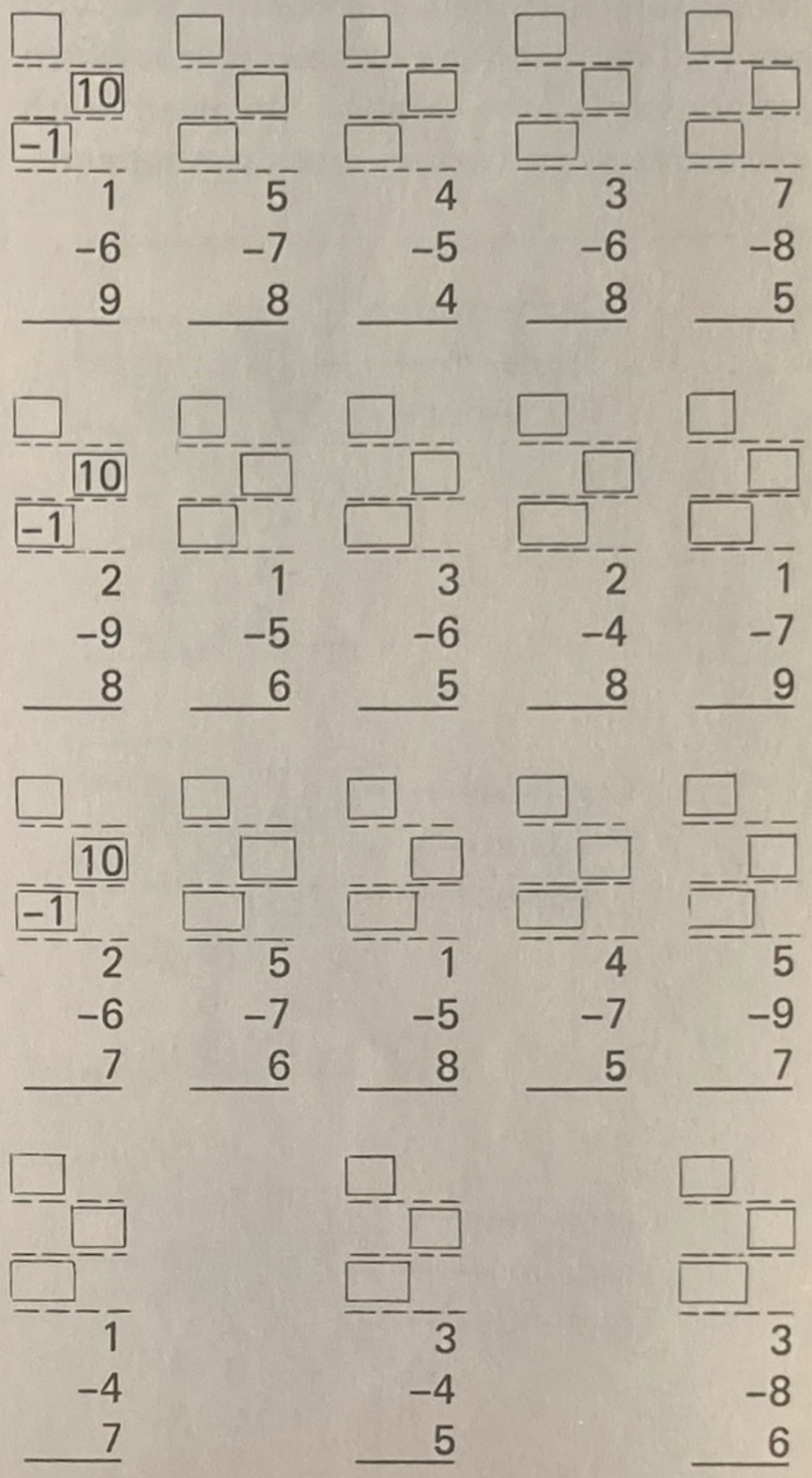




## MIXED OPERATIONS

Now that you are familiar with Consistent HYPAI Trading, you can use this technique to calculate problems of various natures. One type of problem that is not usually found in conventional math books is what we call mixed operations where there are two positive numbers and one subtrahend. Consider the problem: 3 - 7 + 8 = ? In conventional math, this problem can only be solved by either changing the order of the operations, i.e. 3 + 8 - 7 = ?, or by using negative numbers. However, if you use Consistent HYPAI Trading, you can proceed directly. You will notice from the illustration that we have slightly changed the format of the trade insofar as we have moved all of the trading boxes to the top of the problem. This was done because we are no longer working with one minuend and one subtrahend and also to prepare you for even more complicated problems. As you can see from the illustration, once a Consistent HYPAI Trade is made the solution to this problem is very easy. You first press the 10 from the trade and then the 3 from the first addend; then you clear the 7 from the subtrahend and press the 8 from the second addend. Your fingers should now be reading 14. A 4 is placed below the line and a 1 goes into the carry-over box. Since the numbers in the carry-over box and the trade-out box cancel, your answer is 4.

## PRACTICE EXERCISES

With the new format placing both trading boxes at the top of the problem, special care must be taken that the students are placing the numbers in the proper spaces.



## MULTIPLE SUBTRAHENDS

Consistent HYPAI Trading can also be used for problems with multiple subtrahends. However, two format adjustments must be made. As in the mixed operations problems, we place all the trade boxes at the top of the problems. Now the rules for Consistent HYPAI Trading are: 1) Make a trade for *each* subtrahend. If there are 2 subtrahends, you must be trading 20's. If there are three subtrahends, you must be trading 30's. 2) Make a trade in *every column* of the problem. Make sure it is the same number that you are trading. See the examples below which demonstrate the trades involved with problems with two or more subtrahends.

Carry-over → □
Trade-in → 20
Trade-out → -2

```
  9
 -3
 -4
  5
```

Carry-over → □□
Trade-in → 30 30
Trade-out → -3 -3

```
  8 9
 -1 5
 -2 2
 -  7
```

Carry-over → □□
Trade-in → 20 20
Trade-out → -2 -2

```
  7 5
 -  6
 -2 9
```

Carry-over → □□□
Trade-in → 20 20 20
Trade-out → -2 -2 -2

```
 -2 0 8
  8 6 3
 -  7 9
```

## PRACTICE EXERCISES

Since there are two subtrahends, 2 tens must be traded out of the tens' column into the units' column. If the students do not see where the 20 is coming from, explain it as first adding 20 and then subtracting 20 from the same set of numbers. If you add and then subtract the same number, you do not change the value of the problem.

| | | | | |
|---|---|---|---|---|
| □ | □ | □ | □ | □ |
| 20 | □ | □ | □ | □ |
| -2 | □ | □ | □ | □ |
| 7 | 5 | 2 | 1 | 4 |
| -9 | -8 | -5 | -6 | -7 |
| 8 | 6 | 7 | 8 | 5 |
| -2 | -3 | -1 | -2 | -1 |

| | | | | |
|---|---|---|---|---|
| □ | □ | □ | □ | □ |
| 20 | □ | □ | □ | □ |
| -2 | □ | □ | □ | □ |
| 8 | 5 | 3 | 6 | 5 |
| -9 | -7 | -4 | -8 | -6 |
| -3 | -6 | -5 | -4 | -2 |
| 9 | 9 | 8 | 7 | 6 |

A trade must be made into every column of the problem. For consistency, the same number should be traded. On this page, you should trade 2 tens for a 20 in the units' column and 2 hundreds for a 20 in the tens' column.

| | | |
|---|---|---|
| 20 20<br>-2 -2<br>7 3<br>-1 9<br>- 8 | 8 1<br>-2 7<br>- 4 | 9 5<br>- 8<br>-3 7 |
| 20 20<br>-2 -2<br>- 7<br>-5 6<br>8 0 | - 6<br>-1 8<br>9 3 | -2 4<br>- 9<br>5 2 |
| 20 20<br>-2 -2<br>- 6<br>7 5<br>-1 7 | - 9<br>5 4<br>-2 7 | -4 3<br>7 8<br>- 2 |
| 8 4<br>- 6<br>-4 5 | -3 6<br>- 7<br>8 3 | -2 7<br>8 9<br>- 3 |

Care must be taken to press the addends and clear the subtrahends. When the numbers are moved about in an untraditional manner, students can make mistakes—more from carelessness than from lack of understanding.

| | | |
|---|---|---|
| 20 20 20<br>-2 -2 -2<br>9 0 4<br>- 9<br>-1 9 5 | 7 5 0<br>- 9 5<br>-3 4 7 | 8 1 3<br>- 9 9<br>- 7 7 |
| 20 20 20<br>-2 -2 -2<br>- 6 9<br>7 9 0<br>- 7 6 | -2 7 8<br>5 1 6<br>-1 0 0 | -1 9 6<br>8 5 9<br>- 3 1 |

| | | |
|---|---|---|
| 20 20 20<br>-2 -2 -2<br>- 9<br>- 8 7<br>6 0 0 | - 7 5<br>- 4<br>9 0 8 | - 4 2<br>- 6 9<br>5 1 3 |

Since there are three subtrahends, a 30 is traded for every column of the problem. Make sure the students are placing the numbers properly for the trade and carry-over boxes.

| | | |
|---|---|---|
| 30 30 30 30<br>-3 -3 -3 -3<br>7,8 6 0<br>-1,9 5 7<br>-2,1 3 6<br>- 9 8 | 8,5 0 4<br>-1,6 3 2<br>-2,8 9 7<br>- 9 5 | |
| 30 30 30 30<br>-3 -3 -3 -3<br>- 9 5 3<br>-4,8 7 6<br>9,0 3 7<br>-1,6 4 5 | -3,0 6 7<br>-4,2 0 4<br>- 5 1 8<br>9,6 2 1 | |
| 30 30 30 30<br>-3 -3 -3 -3<br>- 5 0 3<br>-2,8 2 6<br>- 4 9<br>8,7 3 0 | 9,0 2 7<br>- 4 9<br>-4,7 3 1<br>- 5 9 6 | |
| 7,6 9 3<br>- 8 4<br>-2,7 6 0<br>-2,0 3 2 | -2,5 3 4<br>8,9 5 2<br>- 7 3 5<br>-3,8 1 0 | |

## HYPAI POWER TRADING

We now approach the next to last step in your program of increased speed in the calculation of subtraction. The last step occurs when you make the HYPAI Trades mentally. Before you go on to that step, you must make one more advance which we call POWER TRADING.

POWER TRADING merely combines the HYPAI Trade numbers. Thus instead of writing a series of tens and -1's, Power Trading combines them. The rule is very simple. The units' column receives a 10, every other column of the minuend receives a 9, and a -1 is placed one column to the left of the last digit in the minuend. You may find out that some of your brighter students are already doing this.

(I) CONSISTENT FORMAT (A)

Make a trade for every column of the subtrahend.

```
  □□□
 10 10 10
39,654
-1 -1 -1
-   718
```

(II) CONSISTENT FORMAT (B)

Make a trade for every column of the minuend.

```
  □□□□□
 10 10 10 10 10
-1 -1 -1 -1 -1
 39,654
-   718
```

(III) POWER FORMAT

Combine the trading numbers.

```
 □  □  □  □  □
(-1)(9)(9)(9)(10)
   3, 9  6  5  4
-              7  1  8
```

## PRACTICE EXERCISES

Combine the trading numbers as shown in the first column. The carry-over boxes continue to be shown until they get used to the new format. However, they can be abandoned and the hand transfer used as soon as the students feel comfortable.

```
 1  0  0  0
(-1)(9)(9)(9)(10)
   2, 0  0  0
-              9
```

```
 □  □  □  □
( )( )( )( )( )
   8, 0  0  0
-              7
```

```
 1  0  0  0
(-1)(9)(9)(9)(10)
   5, 0  0  0
-              4
```

```
 □  □  □  □
( )( )( )( )( )
   6, 0  0  0
-              8
```

```
 1  0  0  0
(-1)(9)(9)(9)(10)
   7, 0  0  0
-              3
```

```
 □  □  □  □
( )( )( )( )( )
   5, 0  0  0
-              2
```

```
 1  0  0  0
(-1)(9)(9)(9)(10)
   3, 0  0  0
-              7
```

```
 □  □  □  □
( )( )( )( )( )
   9, 0  0  0
-              4
```

```
 □  □  □  □
( )( )( )( )( )
   3, 0  0  0
-              6
```

```
 □  □  □  □
( )( )( )( )( )
   9, 0  0  0
-              5
```

```
 □  □  □  □
( )( )( )( )( )
   8, 0  0  0
-              9
```

```
 □  □  □  □
( )( )( )( )( )
   1, 0  0  0
-              9
```

This format focuses on the minuend. A trade is made for every digit. Using this format becomes essential later on when there is more than one subtrahend.

| 1 | 0 | 0 | 1 | | |
|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) | |
| | 2, | 0 | 0 | 8 | |
| – | | | 6 | 4 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 8, | 0 | 0 | 9 | |
| – | | | 2 | 7 | |

| 1 | 0 | 0 | 1 | | |
|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) | |
| | 1, | 0 | 0 | 6 | |
| – | | | 8 | 4 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 6, | 0 | 0 | 8 | |
| – | | | 3 | 5 | |

| 1 | 0 | 0 | 1 | | |
|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) | |
| | 7, | 0 | 0 | 5 | |
| – | | | 4 | 2 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 5, | 0 | 0 | 4 | |
| – | | | 9 | 3 | |

| 1 | 0 | 0 | 1 | | |
|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) | |
| | 9, | 0 | 0 | 9 | |
| – | | | 5 | 1 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 1, | 0 | 0 | 6 | |
| – | | | 2 | 3 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 3, | 0 | 0 | 7 | |
| – | | | 1 | 6 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 4, | 0 | 0 | 3 | |
| – | | | 5 | 1 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 2, | 0 | 0 | 7 | |
| – | | | 1 | 6 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 6, | 0 | 0 | 5 | |
| – | | | 7 | 2 | |

You should be aiming to have the entire operation internalized, i.e. the power numbers should eventually be used mentally and the carry-over numbers transferred from the left to the right hand. Eventually the students should be able to calculate within the traditional format.

| 1 | 0 | 0 | 0 | | |
|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) | |
| | 7, | 0 | 0 | 0 | |
| | –2, | 7 | 8 | 5 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 5, | 1 | 3 | 6 | |
| | –3, | 9 | 4 | 8 | |

| 1 | 0 | 0 | 1 | | |
|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) | |
| | 6, | 2 | 4 | 9 | |
| | –1, | 8 | 9 | 0 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 4, | 5 | 7 | 2 | |
| | –1, | 6 | 8 | 1 | |

| 1 | 1 | 0 | 0 | | |
|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) | |
| | 8, | 5 | 0 | 1 | |
| | –6, | 2 | 3 | 2 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 7, | 6 | 3 | 0 | |
| | –1, | 2 | 4 | 9 | |

| 1 | 0 | 1 | 0 | | |
|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) | |
| | 6, | 0 | 7 | 1 | |
| | –2, | 8 | 4 | 9 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 5, | 1 | 8 | 2 | |
| | –3, | 9 | 0 | 4 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 8, | 1 | 0 | 4 | |
| | –6, | 3 | 1 | 7 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 9, | 0 | 1 | 8 | |
| | –7, | 2 | 3 | 4 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 2, | 3 | 5 | 1 | |
| | –1, | 0 | 7 | 8 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 8, | 7 | 4 | 5 | |
| | –5, | 2 | 1 | 0 | |

In problems such as those presented here students will begin to take shortcuts. They will discover that as soon as they reach a 1 for a carry-over their computation can be completed by "bringing down" the rest of the minuend (on this page, the thousands' column).

| 1 | 1 | 0 | 0 | | |
|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) | |
| | 2, | 4 | 0 | 6 | |
| – | | | | 9 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 8, | 9 | 0 | 5 | |
| – | | | | 7 | |

| 1 | 1 | 0 | 0 | | |
|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) | |
| | 5, | 1 | 0 | 0 | |
| – | | | | 4 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 6, | 2 | 0 | 4 | |
| – | | | | 8 | |

| 1 | 1 | 0 | 0 | | |
|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) | |
| | 7, | 6 | 0 | 2 | |
| – | | | | 3 | |

| □ | □ | □ | □ | | |
|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | |
| | 5, | 6 | 0 | 1 | |
| – | | | | 2 | |

| 1 | 1 | 0 | 0 |
|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) |
| | 3, | 4 | 0 | 6 |
| – | | | | 7 |

| □ | □ | □ | □ |
|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| | 9, | 1 | 0 | 2 |
| – | | | | 4 |

| □ | □ | □ | □ | □ | □ | □ |
|---|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) |
| | 2, | 0 | 0 | 8, | 0 | 0 | 0 |
| – | | | | | | | 7 |

| □ | □ | □ | □ |
|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| | 3, | 7 | 0 | 0 |
| – | | | | 6 |

| □ | □ | □ | □ |
|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| | 9, | 2 | 0 | 1 |
| – | | | | 5 |

| □ | □ | □ | □ | □ | □ | □ |
|---|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) |
| | 4, | 0 | 0 | 1, | 0 | 0 | 0 |
| – | | | | | | | 4 |

| □ | □ | □ | □ |
|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| | 8, | 5 | 0 | 0 |
| – | | | | 1 |

| □ | □ | □ | □ |
|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| | 1, | 2 | 0 | 5 |
| – | | | | 9 |

| □ | □ | □ | □ | □ | □ | □ |
|---|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) |
| | 6, | 0 | 0 | 4, | 0 | 0 | 0 |
| – | | | | | | | 1 |

| □ | □ | □ | □ | □ | □ | □ |
|---|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) |
| | 8, | 0 | 0 | 5, | 0 | 0 | 0 |
| – | | | | | | | 3 |

In principle, the Power Trade should be made for every column. In practice the students will stop once they have found a 1 for a carry-over, (here in the thousands' column). They will "bring down" the last three digits of the minuend.

| □ | □ | □ | 1 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (9) | (9) | (9) | (10) |
| | 1, | 0 | 0 | 2, | 0 | 0 | 0 |
| – | | | | | | | 9 |

| □ | □ | □ | 1 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (9) | (9) | (9) | (10) |
| | 3, | 0 | 0 | 3, | 0 | 0 | 0 |
| – | | | | | | | 6 |

| □ | □ | □ | 1 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (9) | (9) | (9) | (10) |
| | 5, | 0 | 0 | 6, | 0 | 0 | 0 |
| – | | | | | | | 5 |

| □ | □ | □ | 1 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (9) | (9) | (9) | (10) |
| | 7, | 0 | 0 | 7, | 0 | 0 | 0 |
| – | | | | | | | 2 |

This kind of problem has many practical applications in every day life, but it is not found in conventional textbooks because they aren't able to handle them.

| 1 | □ | □ | □ | □ |
|---|---|---|---|---|
| (–1)(10) | ( ) ( ) | ( ) ( ) | ( ) ( ) | ( ) ( ) |
| 2 | 6 | 3 | 1 | 3 |
| –7 | –7 | –5 | –4 | –6 |
| 8 | 5 | 8 | 6 | 7 |

| 1 | □ | □ | □ | □ |
|---|---|---|---|---|
| (–1)(10) | ( ) ( ) | ( ) ( ) | ( ) ( ) | ( ) ( ) |
| 1 | 4 | 5 | 3 | 2 |
| –5 | –7 | –6 | –4 | –6 |
| 6 | 4 | 8 | 5 | 5 |

| 1 | □ | □ | □ | □ |
|---|---|---|---|---|
| (–1)(10) | ( ) ( ) | ( ) ( ) | ( ) ( ) | ( ) ( ) |
| 2 | 7 | 7 | 5 | 3 |
| –5 | –9 | –8 | –7 | –8 |
| 9 | 8 | 5 | 4 | 9 |

| □ | □ | □ |
|---|---|---|
| ( )( ) | ( )( ) | ( )( ) |
| 5 | 8 | 4 |
| −9 | −9 | −6 |
| 8 | 3 | 5 |

This type of problem is excellent for instilling in young students increased insight into the nature of addition and subtraction. The inverse relationship of the two operations quickly becomes apparent.

| 1 | □ | □ | □ |
|---|---|---|---|
| (−1)(10) | ( )( ) | ( )( ) | ( )( ) |
| 3 | 2 | 4 | 5 |
| −4 | −3 | −5 | −6 |
| 5 | 4 | 6 | 7 |

| 1 | □ | □ | □ |
|---|---|---|---|
| (−1)(10) | ( )( ) | ( )( ) | ( )( ) |
| 7 | 3 | 1 | 2 |
| −8 | −4 | −5 | −8 |
| 9 | 2 | 6 | 7 |

| 1 | □ | □ | □ |
|---|---|---|---|
| (−1)(10) | ( )( ) | ( )( ) | ( )( ) |
| 6 | 4 | 1 | 3 |
| −7 | −5 | −7 | −5 |
| 9 | 8 | 8 | 7 |

| □ | □ | □ |
|---|---|---|
| ( )( ) | ( )( ) | ( )( ) |
| 6 | 5 | 5 |
| −7 | −7 | −6 |
| 8 | 6 | 8 |

Because there are two subtrahends, the Power Trade numbers are 20 and −2. Since the subtrahends are found in different positions, special care must be shown to *press* addends and *clear* the subtrahends.

| 2 | □ | □ | □ |
|---|---|---|---|
| (−2)(20) | ( )( ) | ( )( ) | ( )( ) |
| 6 | 5 | −4 | 9 |
| −2 | −3 | 7 | −3 |
| −4 | 7 | −3 | −4 |
| 9 | −5 | 8 | 5 |

| 2 | □ | □ | □ |
|---|---|---|---|
| (−2)(20) | ( )( ) | ( )( ) | ( )( ) |
| 3 | −1 | 7 | −6 |
| 7 | 6 | −2 | 8 |
| −6 | −8 | −1 | −3 |
| −2 | 9 | 5 | 4 |

| 2 | □ | □ | □ |
|---|---|---|---|
| (−2)(20) | ( )( ) | ( )( ) | ( )( ) |
| −4 | 6 | −7 | 5 |
| 5 | −3 | 8 | −2 |
| −3 | −2 | −4 | −4 |
| 7 | 1 | 8 | 3 |

| □ | □ | □ |
|---|---|---|
| ( )( ) | ( )( ) | ( )( ) |
| −2 | 9 | 8 |
| 1 | −4 | −5 |
| 9 | 3 | 3 |
| −4 | −5 | −1 |

Eventually, the numbers from the Power Trade can be placed in the problem mentally and do not have to be written down. This step should be taken only after placing the numbers from the Power Trade is automatic and completely accurate.

| 2 | □ | □ | □ |
|---|---|---|---|
| (−2)(20) | ( )( ) | ( )( ) | ( )( ) |
| 7 | −8 | −3 | 9 |
| −5 | 7 | −4 | −6 |
| −4 | −1 | 6 | 4 |
| 6 | 9 | 7 | −5 |

| 2 | □ | □ | □ |
|---|---|---|---|
| (−2)(20) | ( )( ) | ( )( ) | ( )( ) |
| −2 | 3 | −8 | 4 |
| −8 | −6 | 7 | −2 |
| 6 | 9 | −5 | −1 |
| 5 | −3 | 9 | 7 |

| 2 | □ | □ | □ |
|---|---|---|---|
| (−2)(20) | ( )( ) | ( )( ) | ( )( ) |
| 7 | −5 | 8 | −3 |
| 4 | 9 | −9 | −7 |
| −3 | −6 | 9 | 2 |
| −2 | 7 | −4 | 8 |

| □ | □ | □ |
|---|---|---|
| ( )( ) | ( )( ) | ( )( ) |
| 5 | 5 | 5 |
| 9 | 6 | −1 |
| −2 | −2 | 4 |
| −4 | −8 | −6 |

With two subtrahends and two columns of numbers, the Power Trade number in the tens' column is 18. Even though some of the subtrahends are one-digit numbers, we trade the same numbers in each column for the sake of consistency.

| □ □ | □ □ | □ □ |
|---|---|---|
| (−2)(18)(20) | ( )( )( ) | ( )( )( ) |
| 4 0 | 5 6 | 6 4 |
| −1 2 | − 8 | −1 3 |
| −1 5 | −1 1 | −2 2 |

| □ □ | □ □ | □ □ |
|---|---|---|
| (−2)(18)(20) | ( )( )( ) | ( )( )( ) |
| 5 1 | 6 4 | 7 5 |
| − 5 | −2 7 | − 6 |
| −2 6 | −1 6 | −2 9 |

| □ □ | □ □ | □ □ |
|---|---|---|
| (−2)(18)(20) | ( )( )( ) | ( )( )( ) |
| 8 6 | 9 7 | 6 7 |
| −2 5 | − 9 | −1 8 |
| −1 9 | −2 8 | −3 5 |

| □ □ | □ □ | □ □ |
|---|---|---|
| ( )( )( ) | ( )( )( ) | ( )( )( ) |
| 7 5 | 8 1 | 8 3 |
| − 5 | −3 2 | −2 4 |
| −3 3 | −1 6 | −3 8 |

| □ □ | □ □ | □ □ |
|---|---|---|
| ( )( )( ) | ( )( )( ) | ( )( )( ) |
| 9 2 | 7 5 | 5 3 |
| − 6 | − 7 | −1 8 |
| −3 6 | −2 4 | −1 9 |

Since the minuend has three digits, we must make three Power Trades. Hence the trade number in the tens' and hundreds' column is 18.

| □ □ □ | □ □ □ |
|---|---|
| (−2)(18)(18)(20) | ( )( )( )( ) |
| 2 0 0 | − 6 |
| − 4 2 | − 5 3 |
| − 7 | 9 1 4 |

| □ □ □ | □ □ □ |
|---|---|
| (−2)(18)(18)(20) | ( )( )( )( ) |
| 6 1 5 | − 8 |
| − 9 | 7 1 6 |
| − 5 2 | − 4 3 |

| □ □ □ | □ □ □ |
|---|---|
| (−2)(18)(18)(20) | ( )( )( )( ) |
| 5 1 | 6 4 |
| − 5 | −2 7 |
| −2 6 | −1 6 |

| □ □ □ | □ □ □ |
|---|---|
| (−2)(18)(18)(20) | ( )( )( )( ) |
| 6 9 8 | 5 2 6 |
| − 7 6 | − 9 3 |
| − 5 7 | − 7 1 |

| | | | |
|---|---|---|---|
| □□□<br>( )( )( )( )<br>– 1 6<br>– 9 2<br>7 0 5 | □□□<br>( )( )( )( )<br>– 4 6<br>8 1 7<br>– 6 3 | □□□<br>( )( )( )( )<br>– 7<br>– 5<br>5 0 6<br>1 2 | □□□<br>( )( )( )( )<br>– 9<br>3 0 1<br>– 8<br>1 6 |
| □□□<br>( )( )( )( )<br>– 1 8<br>7 3 0<br>– 7 2 | □□□<br>( )( )( )( )<br>8 0 1<br>– 4 9<br>– 5 | □□□<br>( )( )( )( )<br>4 0 6<br>– 9<br>– 8<br>5 1 | □□□<br>(–2) (18) (18) (20)<br>9 1 4<br>– 6<br>– 9 1<br>2 0 |
| □□□<br>( )( )( )( )<br>– 7 2<br>– 9 5<br>3 6 1 | □□□<br>( )( )( )( )<br>9 8 6<br>– 2 4<br>– 1 3 | □□□<br>( )( )( )( )<br>– 7<br>7 8<br>4 0 5<br>– 9 9 | □□□<br>( )( )( )( )<br>3 5<br>– 8 8<br>– 4<br>6 5 3 |

The carry-over boxes are placed here to maintain the format. If the students are capable of transferring the carry-over number from the left hand to the right, they do not have to write the carry-over number.

| | | | |
|---|---|---|---|
| □□□<br>(–2) (18) (18) (20)<br>5 0 0<br>– 4<br>– 9 | □□□<br>( )( )( )( )<br>8 0 7<br>– 8<br>– 5 | □□□<br>( )( )( )( )<br>– 4 7<br>5 3 2<br>1 1<br>– 8 | □□□<br>( )( )( )( )<br>7 0<br>– 3<br>– 9 7<br>4 2 2 |
| □□□<br>( )( )( )( )<br>– 4<br>– 9<br>4 0 2 | □□□<br>(–2) (18) (18) (20)<br>1 0<br>– 8<br>– 9<br>6 0 2 | □□□<br>( )( )( )( )<br>– 4<br>6 1<br>– 8 7<br>5 0 2 | □□□<br>(–2) (18) (18) (20)<br>1 0 3<br>4 5 0<br>– 6 2<br>– 4 |
| □□□<br>( )( )( )( )<br>1 9<br>– 7<br>4 0 3<br>– 8 | □□□<br>( )( )( )( )<br>3 0 8<br>1 2<br>– 6<br>– 7 | □□□<br>( )( )( )( )<br>– 7 6<br>3 6 0<br>– 9<br>6 0 5 | □□□<br>( )( )( )( )<br>– 8<br>2 0 7<br>5 4 0<br>– 5 4 |

| □ □ □ | □ □ □ |
|---|---|
| ( ) ( ) ( ) ( ) | ( ) ( ) ( ) ( ) |
| 7 0 2 | 5 1 0 |
| – 1 | – 7 2 |
| 2 3 0 | – 6 |
| – 4 5 | 3 0 4 |

| □ □ □ □ | □ □ □ □ |
|---|---|
| (–2) (18) (18) (18) (20) | ( ) ( ) ( ) ( ) ( ) |
| 6, 0 2 7 | 8, 5 3 4 |
| – 7 6 8 | – 4 6 |
| – 9 3 | – 8 9 7 |

As you continue the Power Trade format, it is useful to occasionally review why the middle columns have 18. Remember two tens were traded into these columns and two ones were traded out.

| □ □ □ | □ □ □ |
|---|---|
| (–2) (18) (18) (20) | ( ) ( ) ( ) ( ) |
| 4 8 6 | 7 0 0 |
| –1 3 9 | –4 2 5 |
| –1 7 8 | –1 0 6 |

| □ □ □ | □ □ □ |
|---|---|
| (–2) (18) (18) (20) | ( ) ( ) ( ) ( ) |
| –2 1 3 | –1 9 3 |
| 9 0 2 | 6 0 6 |
| –3 8 7 | –2 1 0 |

| □ □ □ | □ □ □ |
|---|---|
| (–2) (18) (18) (20) | ( ) ( ) ( ) ( ) |
| –5 7 8 | – 2 3 |
| – 1 8 | –4 5 6 |
| 9 0 0 | 9 0 3 |

| □ □ □ | □ □ □ |
|---|---|
| ( ) ( ) ( ) ( ) | ( ) ( ) ( ) ( ) |
| 9 4 7 | –2 0 8 |
| –3 5 9 | 8 6 3 |
| –1 6 8 | –2 7 9 |

Since the minuend has four digits, four Power Trades must be made. As in all the other examples, the middle columns show the number 18.

| □ □ □ □ | □ □ □ □ |
|---|---|
| (–2) (18) (18) (18) (20) | ( ) ( ) ( ) ( ) ( ) |
| 9, 0 3 2 | 7, 1 4 3 |
| – 7 5 | – 5 8 |
| – 4 6 | – 6 9 |

| □ □ □ □ | □ □ □ □ |
|---|---|
| (–2) (18) (18) (18) (20) | ( ) ( ) ( ) ( ) ( ) |
| 4, 2 1 6 | –3, 9 0 7 |
| –2, 0 6 4 | 7, 3 4 5 |
| – 9 2 | – 6 1 |

| □ □ □ □ | □ □ □ □ |
|---|---|
| ( ) ( ) ( ) ( ) ( ) | ( ) ( ) ( ) ( ) ( ) |
| 9, 6 1 0 | – 2 7 |
| – 5 7 | – 5 8 |
| – 4 8 | 1, 6 4 9 |

When there are three subtrahends, the Power Trade numbers become 30 in the units' column, 27 in the middle columns, and –3 in the column to the left of the last digit in the minuend.

| □ □ □ □ | □ □ □ □ |
|---|---|
| (–3) (27) (27) (27) (30) | ( ) ( ) ( ) ( ) ( ) |
| 9, 0 4 3 | 7, 2 3 1 |
| –2, 7 8 6 | – 7 5 |
| – 5 2 7 | – 9 6 4 |
| – 6 0 | –1, 8 0 5 |

| □ | □ | □ | □ | | | □ | □ | □ | □ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (–3) | (27) | (27) | (27) | (30) | | ( ) | ( ) | ( ) | ( ) | ( ) | |
| | – | | | 4 | 6 | | – | | 7 | 2 | 6 |
| | | 7, | 0 | 8 | 9 | | | –2, | 8 | 5 | 9 |
| | – | | 5 | 2 | 7 | | | | 8, | 9 | 0 | 3 |
| | | –1, | 6 | 3 | 0 | | – | | | 1 | 4 |

| □ | □ | □ | □ | | | □ | □ | □ | □ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (–3) | (27) | (27) | (27) | (30) | | ( ) | ( ) | ( ) | ( ) | ( ) | |
| | | –6, | 9 | 4 | 3 | | | –2, | 6 | 9 | 3 |
| | – | | 8 | 5 | 2 | | | | 8, | 5 | 0 | 4 |
| | | 9, | 0 | 1 | 6 | | – | | | 7 | 1 |
| | – | | | 9 | 5 | | – | 2 | 1 | 8 | |

| □ | □ | □ | □ | | | □ | □ | □ | □ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | | ( ) | ( ) | ( ) | ( ) | ( ) | |
| | | 6, | 9 | 0 | 4 | | | –1, | 2 | 4 | 9 |
| | – | | 2 | 1 | 3 | | – | | | 7 | 5 |
| | – | | | 5 | 9 | | – | | 6 | 0 | 8 |
| | | –3, | 0 | 4 | 7 | | | 9, | 1 | 3 | 0 |

## MULTIDIGIT MISSING SUBTRAHEND

To find a missing subtrahend that is more than one digit is rather simple under the Power Trading format. For clarity, we will continue to use the carry-over boxes, but these can be discarded if you are using hand transfers. In the example below you write the Power Trade numbers first.

| □ | □ | □ | □ | ← Carry-over |
|---|---|---|---|---|
| (–1) | (9) | (9) | (10) | ← Power Trades |
| 7 | 2 | 3 | 6 | |
| – □ | □ | □ | □ | |
| 2 | 7 | 1 | 9 | |

To find the units' number of the subtrahend, you must press the 10 from the trade and the 6 from the minuend. Then, counting as you clear, you must clear your right hand until it shows the 9 found in the units' column of the difference. The number you counted, in this case 7, is the missing units' digit. Since there is nothing registered on your left hand, a 0 is placed into the carry-over box in the tens' column.

| □ | □ | □ | 0 | |
|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) |
| 7 | 2 | 3 | 6 | |
| – □ | □ | □ | 7 | |
| 2 | 7 | 1 | 9 | |

To find the missing digit in the tens' column, you press the 9 from the Power Trade and another 3 from the minuend. While counting, you clear until your right hand shows the 1 in the difference. In this case you had to clear a 1. Since there is a 10 registered on your left hand, you must place a 1 in the carry-over box in the hundreds' column or transfer it directly to your right hand.

| □ | □ | 1 | 0 | |
|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) |
| 7 | 2 | 3 | 6 | |
| – □ | □ | 1 | 7 | |
| 2 | 7 | 1 | 9 | |

To find the missing digit in the hundreds' column, press the 1 from the carry-over, the 9 from the Power Trade and the 2 from the minuend. Clear until there is a 7 on your right hand. Counting as you clear, you find the missing digit to be 5. Since there is nothing on the left

hand, a 0 is placed into the carry-over box in the thousands' column.

| □ | 0 | 1 | 0 | |
|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) |
| | 7 | 2 | 3 | 6 |
| – | □ | 5 | 1 | 7 |
| | 2 | 7 | 1 | 9 |

The last missing digit is found by pressing the 9 from the Power Trade and the 7 from the minuend. You then clear until your right hand shows the 2 that is in the difference. Counting as you clear you find the missing digit to be a 4. Since there is a 10 on your left hand, this is carried over to the next column where it cancels with the –1 from the Power Trade.

| 1 | 0 | 1 | 0 | |
|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) |
| | 7 | 2 | 3 | 6 |
| – | 4 | 5 | 1 | 7 |
| | 2 | 7 | 1 | 9 |

## PRACTICE EXERCISES

To find the missing digit of the subtrahend you must press the carry-over and trade numbers. Press the minuend; then *clear* until your right-hand fingers show the number below the line in the difference. In the last three rows where there is a digit missing in the difference, press the carry-over, trade and minuend numbers and then clear the digit of the subtrahend to find the missing digit in the difference.

The digits missing in the subtrahends are found by counting orally as you clear. The digit missing in the difference is found on the fingers.

| □ | □ | □ | □ | |
|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) |
| | 6, | 2 | 3 | 5 |
| – | □ | □ | □ | □ |
| | 3, | 0 | 6 | 0 |

| □ | □ | □ | □ | |
|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| | 7, | 4 | 9 | 3 |
| – | □ | □ | □ | □ |
| | 2, | 7 | 0 | 8 |

| □ | □ | □ | □ | |
|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) |
| | 4, | 0 | 1 | 6 |
| – | □ | □ | □ | □ |
| | 1, | 9 | 3 | 2 |

| □ | □ | □ | □ | |
|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| | 5, | 8 | 0 | 7 |
| – | □ | □ | □ | □ |
| | 2, | 4 | 8 | 6 |

| □ | □ | □ | □ | |
|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| | 8, | 0 | 7 | 6 |
| – | □ | □ | □ | □ |
| | 3, | 5 | 9 | 1 |

| □ | □ | □ | □ | |
|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| | 6, | 2 | 1 | 3 |
| – | □ | □ | □ | □ |
| | 3, | 5 | 9 | 0 |

| □ | □ | □ | □ | |
|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) |
| | 9, | 4 | 8 | 0 |
| – | □ | □ | □ | □ |
| | 5, | 9 | 7 | 3 |

| □ | □ | □ | □ | |
|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| | 7, | 6 | 3 | 5 |
| – | □ | □ | □ | □ |
| | 2, | 5 | 1 | 8 |

Digits missing in both subtrahend and the difference.

| □ | □ | □ | □ | |
|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) |
| – | □ | □ | □ | 9 |
| | 7, | 0 | 9 | 3 |
| | 2, | 7 | 0 | □ |

| □ | □ | □ | □ | |
|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| – | □ | □ | 2 | □ |
| | 8, | 4 | 7 | 6 |
| | 3, | 9 | □ | 1 |

| □ | □ | □ | □ | |
|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| – | □, | 5 | □ | □ |
| | 5, | 2 | 8 | 0 |
| | 1, | □ | 7 | 5 |

| □ | □ | □ | □ | |
|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) |
| | 6, | 4 | 3 | 8 |
| – | 2, | □ | □ | 9 |
| | □, | 7 | 1 | □ |

| □ | □ | □ | □ | |
|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| | 7, | 2 | 0 | 4 |
| – | □, | 5 | 3 | □ |
| | 2, | □ | □ | 3 |

| □ | □ | □ | □ | |
|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| | 9, | 1 | 5 | 3 |
| – | □, | 8 | □ | □ |
| | 7, | □ | 3 | 0 |

## Multiple Subtrahends— Missing Digits

Problems such as the following can be presented to test whether all of the skills of subtraction have been acquired. If a student can solve these problems, he truly understands the concept of subtraction and how to handle missing subtrahends, HYPAI Power Trading, and carry-over boxes. Students also find this type of problem challenging and enjoyable.

| | | | | |
|---|---|---|---|---|
| [2] | [1] | [1] | [0] | ⟵ Carry-over boxes |
| (–2) | (18) | (18) | (18) | (20) ⟵ Power Trade |
| | 7 | 8 | 3 | 2 Numbers |
| – | [1] | 9 | [4] | 5 |
| – | 4 | [0] | 2 | [8] |
| | 1 | 8 | 5 | 9 |

Since there are two subtrahends, the first step is to make two Power Trades. Thus we have placed 20 in the units' column, 18's in the next 3 columns and –2 in the fifth column. We now examine the problem, column-by-column.

In the units' column, we first press the 20 from the Power Trade and then another 2 from the minuend. We then *clear* a 5 since this digit is part of a subtrahend. At this point, our fingers should read 17. We now must continue to *clear,* since the missing digit is in a subtrahend, until a picture of 9 is reached on our right hand. Counting as we clear, we find we have cleared an additional 8 units. Therefore, 8 goes into the missing digit box. Since there is nothing registered on our left hand, 0 goes into the carry-over box.

Approaching the tens' column, we press 18 because of the Power Trade. We then press an additional 3 from the minuend, and clear –2 because of the known digit in a subtrahend. Our fingers now read 19 and we must continue to clear until our right hand reads 5. Counting as we clear, we find the missing digit is 4. We therefore place a 4 into the missing digit box. Since our left hand registers a 10, we place a 1 into the carry-over box.

To calculate the hundreds' column, we first press a 1 for the carry-over box, and an additional 18 from the Power Trade. Press an additional 8 for the minuend. We clear 9 for the known digit of a subtrahend. Since our fingers read 18, our right hand shows the same number as in the answer below the line. Therefore, we do not have to clear any more numbers in this column. Thus we place a 0 in the missing digit box and a 1 in the carry-over box.

In the thousands' column, we first press a 1 and then an 18 to account for the carry-over and Power Trade boxes. We continue to press an additional 7 as seen in the minuend. We then clear 4 since this is the known digit of a subtrahend. Our fingers now read 22. We must clear one more to have the fingers of our right hand read the same as the answer below the line. Since we had to clear 1, we place a 1 in the missing digit box. Since our left hand reads 20, we place a 2 into the carry-over box.

Finally, we press a 2 from the carry-over box and then clear the –2 from the Power Trade. Since our fingers now read 0, we know that we have completed the problem successfully.

## PRACTICE EXERCISES

Press the numbers from the carry-over, trades and minuends. Clear the known subtrahend. Continue to clear (and count) until your right-hand fingers show the number on the difference. The number that is counted is the missing digit. In the last row, you found the missing digit in the difference by pressing the carry-over, trade, and minuend numbers and then clearing the numbers from the subtrahends. The answer is found on the fingers.

| □ | □ | □ | □ | | □ | □ | □ | □ | |
|---|---|---|---|---|---|---|---|---|---|
| (–2) | (18) | (18) | (18) | (20) | ( ) | ( ) | ( ) | ( ) | ( ) |
| | 6, | 9 | 0 | 4 | | 7, | 5 | 6 | 0 |
| – | 1, | 5 | □ | □ | – | □, | □ | 9 | 3 |
| – | □, | □ | 6 | 8 | – | 1, | 8 | □ | □ |
| | 2, | 0 | 4 | 5 | | 3, | 1 | 5 | 9 |

| □ | □ | □ | □ | | □ | □ | □ | □ | |
|---|---|---|---|---|---|---|---|---|---|
| (–2) | (18) | (18) | (18) | (20) | ( ) | ( ) | ( ) | ( ) | ( ) |
| – | □, | □ | 7 | 0 | – | □, | 5 | 6 | □ |
| | 7, | 0 | 4 | 2 | | 8, | 6 | 0 | 7 |
| – | 2, | 5 | □ | □ | – | 1, | □ | □ | 0 |
| | 1, | 6 | 0 | 9 | | 4, | 3 | 5 | 2 |

| □ | □ | □ | □ | | □ | □ | □ | □ | |
|---|---|---|---|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) |
| | 9, | 8 | 7 | 5 | – | 2, | □ | 8 | □ |
| – | 5, | □ | □ | 4 | | 6, | 2 | 9 | 1 |
| – | □, | 1 | 0 | □ | – | □, | 7 | □ | 5 |
| | 1, | 6 | 7 | 8 | | 1, | 0 | 4 | 9 |

Digits missing in both the subtrahends and the difference.

| □ | □ | □ | □ | | □ | □ | □ | □ | |
|---|---|---|---|---|---|---|---|---|---|
| (–2) | (18) | (18) | (18) | (20) | ( ) | ( ) | ( ) | ( ) | ( ) |
| – | 3, | 2 | □ | □ | – | □, | 7 | 3 | □ |
| – | □, | 8 | 9 | 8 | – | 4, | □ | 7 | 5 |
| | 8, | 9 | 0 | 4 | | 9, | 2 | 3 | 7 |
| | 1, | □ | 9 | 0 | | 3, | 6 | □ | 4 |

### Skills Analysis– Missing Digits (Addends and Subtrahends)

The following problems present a comprehensive test of computational skills in addition and subtraction. If a student can perform these exercises, he then clearly understands the concepts of addition, subtraction, missing addends, missing subtrahends, carry-over numbers, and HYPAI Power Trading.

| □ | □ | □ | □ | ——→ Carry-over |
|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) ——→ HYPAI Power Trades |
| + | 6 | □ | 3 | □ ——→ Missing Digits–Addends |
| – | □ | 7 | □ | 5 ——→ Missing Digits–Subtrahends |
| | 1 | 4 | 8 | 2 |

The first step is to make a Power Trade. We start with the units' column. We first press a 10 from the Power Trade; this is followed by clearing 5 since this digit is part of a subtrahend. Finally, we must *press* (since the missing digit is from the minuend) until a picture of the 2 in the answer below the line is registered on our hands. Counting as we press, we find that we have added 7, and our fingers read 12. We therefore place a 7 in the missing digit box, and a 1 in the carry-over box.

Approaching the tens' column, we press a 1 from the carry-over and a 9 from the Power Trade. We press a 3 because of the known digit in the minuend. Our fingers now read 13. Since the missing digit is part of the subtrahend, we must now *clear*

our fingers until a picture of the 8 in the answer below the line appears on our right hand. Counting as we clear, we find that the missing digit is a 5 which we place in its box. Since there are no tens on our left hand at this point, we place a 0 in the carry-over box.

In the hundreds' column, we press a 9 from the Power Trade. We clear a 7 since this number was part of the subtrahend. Our fingers now read 2. Since the missing digit was in the minuend, we now must *press* until a picture of 4 is reached on our right hand. Counting as we press, we find that a 2 should be placed in the missing digit box. Since our left hand is empty, we place a 0 in the carry-over box.

In the thousands' column, we press a 9 from the Power Trade and then press another 6 from the minuend. Our fingers read 15. Since the missing digit is in the subtrahend, we now must *clear* until a picture of the 1 in the answer below the line is registered on our right hand. Counting as we clear, we find that a 4 should be placed in the missing digit box. Since there is a 10 on our left hand, we have to place a 1 in the carry-over box. This cancels the –1 from the HYPAI Power Trade and we have completed the problem.

This same procedure can be used in problems with multiple addends and multiple subtrahends. The only difference will be making sure to make a Power Trade for each subtrahend. We will now give an example.

| □ | □ | □ | □ | |
|---|---|---|---|---|
| (–2) | (18) | (18) | (18) | (20) |
| + | □ | 5 | 3 | 6 |
| – | 4 | □ | 9 | 2 |
| + | 6 | 0 | □ | 4 |
| – | 2 | 8 | 5 | □ |
| | 4 | 0 | 2 | 1 |

As with the prior example, you calculate this problem, column-by-column.

To calculate the units' column, first press 20 from the Power Trade. Next, press 6 more from the 1st addend. Clear 2 from the 1st subtrahend, and then press another 4 for the 2nd addend. With the fingers reading 28, you must now *clear* until a picture of 1, the number in the answer below the line, appears on your right hand. You *must clear* because the missing digit is from the subtrahend. Counting as you clear, you find that you have cleared 7. Thus a 7 goes into the missing digit box and a 2 goes into the carry-over box since you have a 20 on your left hand.

The tens' column is similar. First press 2 from the carry-over and 18 from the Power Trade. Now press another 3 from the 1st addend, clear 9 from the 1st subtrahend, and another 5 from the 2nd subtrahend. Since the missing digit is from an addend, you *must press* until a picture of the 2 in the answer below the line appears on your right fingers. Counting as you press, you find that the missing digit is 3 and that a 1 goes into the carry-over box.

Proceeding to the hundreds' column, first press 1 from the carry-over box and 18 from the Power Trade. Next, press another 5 from the 1st addend. Since the digit of the 2nd addend is 0, no finger manipulations are made for this number. Now you must clear 8 from the 2nd subtrahend. Next, with your fingers reading 16 you *must clear* your right hand to the number 0 shown in the answer below the line in order to find the missing digit of a subtrahend. Counting as you clear, you find that a 6 goes into the missing digit box and a 1 goes into the carry-over box since you have a 10 registered on your left hand.

Finally, for the thousands, you press a 1 and then an 18 to take into account the carry-over and Power Trade. Then clear a 4 from the 1st subtrahend, press a 6 from the 2nd addend, and clear a 2 from the 2nd subtrahend. Since the missing digit is an addend, you *must press* from the 19 on your fingers until a 4 shows on your right hand. Counting, you find that a 5 goes into the missing digit box, and a 2 into the carry-over box. The 2 in the carry-over box cancels with the –2 from the Power Trade and the problem is completed.

## PRACTICE EXERCISES

Since there are two subtrahends, the trade numbers are 20, 18 and –2. When the missing digit is from the minuend, you must press until your fingers show the digit below the line. If the missing digit is from the subtrahend, you must clear until your fingers show the number below the line. All the known digits must be pressed or cleared before searching for the missing digit.

| | | | | |
|---|---|---|---|---|
| (–2) | (18) | (18) | (18) | (20) |
| + | 9, | 2 | □ | 6 |
| – | 6, | 8 | 7 | 3 |
| + | 7, | □ | 4 | 2 |
| – | □, | 5 | 0 | □ |
| | 5, | 0 | 4 | 2 |

| | | | | |
|---|---|---|---|---|
| (–2) | (18) | (18) | (18) | (20) |
| + | 3, | 5 | □ | 7 |
| + | 7, | 0 | 6 | □ |
| – | 2, | □ | 3 | 5 |
| – | □, | 8 | 7 | 1 |
| | 1, | 6 | 0 | 8 |

| | | | | |
|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| – | 5, | 5 | □ | 7 |
| + | 9, | 0 | 6 | □ |
| + | □, | 6 | 9 | 3 |
| – | 2, | □ | 4 | 9 |
| | 9, | 6 | 8 | 1 |

| | | | | |
|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) |
| + | □, | 7 | □ | 8 |
| – | 2, | □ | 6 | □ |
| | 4, | 2 | 3 | 9 |

| | | | | |
|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) |
| + | 4, | □ | 7 | □ |
| – | □, | 5 | □ | 6 |
| | 1, | 2 | 3 | 4 |

| | | | | |
|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| + | □, | 7 | 9 | 4 |
| + | 4, | □ | 2 | 0 |
| – | 5, | 6 | □ | 9 |
| – | 1, | 3 | 8 | □ |
| | 7, | 1 | 4 | 2 |

| | | | | |
|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| – | 2, | 4 | 3 | □ |
| – | 1, | 3 | □ | 9 |
| + | 6, | □ | 4 | 5 |
| + | □, | 6 | 7 | 8 |
| | 1 1, | 2 | 8 | 9 |

| | | | | |
|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) |
| + | 5, | □ | 0 | 7 |
| – | □, | 7 | 2 | 9 |
| + | 8, | 7 | 1 | □ |
| – | 4, | 5 | □ | 6 |
| | 6, | 0 | 0 | 4 |

Be careful to press when the missing digit is from the minuend and to clear when the missing digit is in the subtrahend. You must count as you press or clear the missing digit since the solution is found by the number that you have counted out. All the known digits must be pressed or cleared before searching for the missing digit.

| | | | | |
|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) |
| + | □, | 1 | □ | 6 |
| – | 3, | □ | 5 | □ |
| | 1, | 5 | 0 | 8 |

| | | | | |
|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) |
| + | □, | □ | 3 | 9 |
| – | 3, | 6 | □ | □ |
| | 6, | 0 | 4 | 0 |

# SUBTRACTION–STAGE 2

Subtraction Stage 2 is the counterpart of Addition Stage 2. As in the first stage the fingers that are manipulated are the same, but the order of the manipulation is exactly the opposite. Hence, as in Stage 1, it is essential that the manipulations for addition be fully understood before attempting those for subtraction.

The numbers are once again broken into groups. For the numbers 1, 2, 3, and 4, the instruction is to clear, one-by-one, as in Subtraction Stage 1.

The number five is subtracted in one of two ways. If there is a 5 pressed, i.e. if the right thumb has been pressed, 5 can be subtracted directly by merely clearing the right thumb. To subtract a 5 when the right thumb is not pressed you Interchange 5 by clearing a 10 and pressing 5 taking care that the last ten that was pressed is always the first ten to be cleared.

The numbers 6–9 are subtracted by first subtracting 5 (either directly by clearing 5 or indirectly by the Interchange 5) and then clearing one-by-one saying six, seven, eight, nine. With young students it is helpful to verbalize the value of the number actually being subtracted. With each finger manipulation say clear five, and clear ONE more, six, and clear ONE more, seven, and clear ONE more, eight, and clear ONE more, nine. The instruction "and clear ONE more" can be dropped as soon as the students recognize the pattern of the operations. Remember the last unit that was pressed is the first unit to be cleared.

The one situation that requires special care is when you have a minuend between 50 and 54 and a subtrahend between 5 and 9. In this situation, a "Big Interchange 5" is required. By Big Interchange 5, we mean that the 50th finger (the left thumb) must be cleared, the other tens' fingers (left index, middle, ring and pinky fingers) must be pressed and the right thumb must be pressed. Again 5 is subtracted by subtracting 50 and adding 45 at the same time.

## Six Key Manipulations

There are six key manipulations which must be mastered in order to learn Subtraction Stage 2. They are 5 - 1, 10 - 1, 9 - 5, 14 - 5, 10 - 6, 15 - 6. The first pair of these crucial manipulations is from Stage 1 and is probably already memorized; the second pair is easy to understand and internalize. However, the third pair 10 - 6 and 15 - 6 presents the most difficulty—especially for adults. They should be practiced frequently. Since they are the opposite of the key manipulations of addition, they will be slightly easier the second time round (See chart page 269).

One very useful practice in learning these manipulations is to practice them together with the manipulations from addition. Thus adding and subtracting the following numbers should become a daily practice: 4 + 1 - 1, 9 + 1 - 1, 4 + 5 - 5, 9 + 5 - 5, 4 + 6 - 6, 9 + 6 - 6. Continual practice of these exercises is very helpful. Performing the key manipulations of Stage 2 Addition just before the manipulations of subtraction will leave the practitioner with a picture of the number he is returning to.

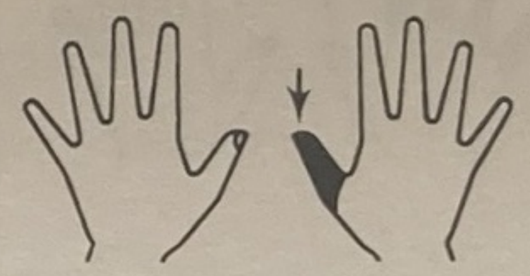
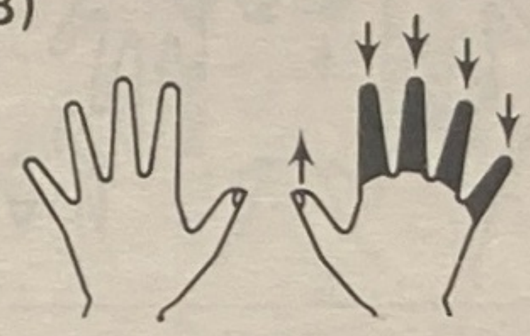
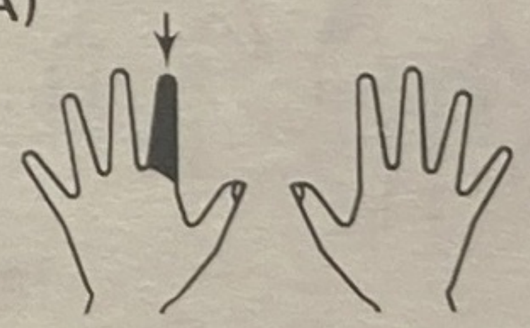
## HOW TO READ NUMBERS (SUBTRAHENDS)

| STAGE 1 SUBTRAHENDS | BREAKDOWN OF SUBTRAHENDS | SAYING: |
|---|---|---|
| –1 | –1 | *ONE* |
| –2 | –1 – 1 | *ONE, TWO* |
| –3 | –1 – 1 – 1 | *ONE, TWO, THREE* |
| –4 | –1 – 1 – 1 – 1 | *ONE, TWO, THREE, FOUR* |
| 1–4 | SUBTRACT UNIT-BY-UNIT | SAYING: *ONE, TWO, THREE, FOUR* |

| STAGE 2 SUBTRAHENDS | BREAKDOWN OF SUBTRAHENDS | SAYING: |
|---|---|---|
| –5 | –5 | *CLEAR FIVE* or *INTERCHANGE FIVE* |
| –6 | –5 – 1 | *CLEAR FIVE* or *INTERCHANGE FIVE* THEN *SIX* |
| –7 | –5 – 1 – 1 | *CLEAR FIVE* or *INTERCHANGE FIVE* THEN *SIX, SEVEN* |
| –8 | –5 – 1 – 1 – 1 | *CLEAR FIVE* or *INTERCHANGE FIVE* THEN *SIX, SEVEN, EIGHT* |
| –9 | –5 – 1 – 1 – 1 – 1 | *CLEAR FIVE* or *INTERCHANGE FIVE* THEN *SIX, SEVEN, EIGHT NINE* |
| 5–9 | SUBTRACT 5 EITHER: CLEAR 5th FINGER or INTERCHANGE 5 (–10 + 5) THEN SUBTRACT UNIT-BY-UNIT. | SAYING: EITHER *CLEAR FIVE* or *INTERCHANGE FIVE* THEN *SIX, SEVEN, EIGHT, NINE* |

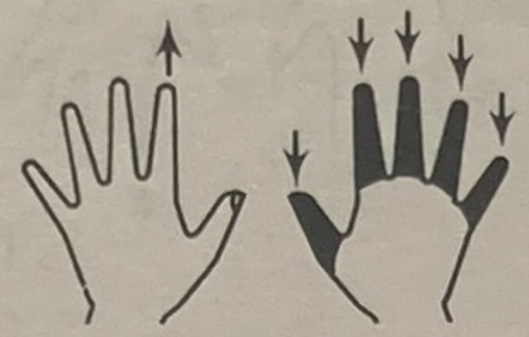
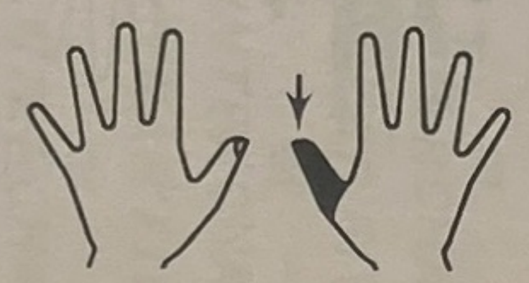
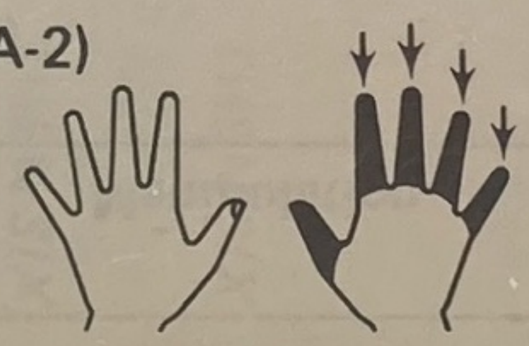
When the 5th finger is AVAILABLE FOR CLEARING, clearing the 5th finger directly to subtract 5.
When the 5th finger is NOT AVAILABLE FOR CLEARING, (Interchange 5) through clearing 10, pressing 5 simultaneously.

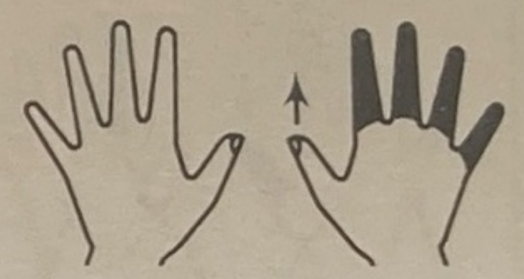
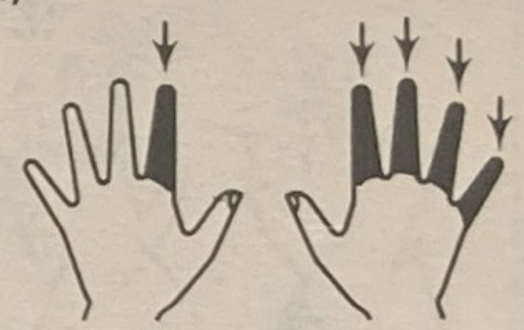
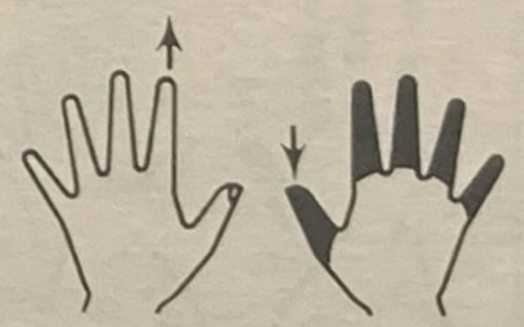
## SIX KEY MANIPULATIONS OF STAGE 2

| Example | Stage | Breakdown of No. | Manipulation | Verbal Count* | Voice Intonation |
|---|---|---|---|---|---|
| 1)<br>A) 5 | A) 2nd | A) 5 | A)  | A) Press five (5) | A) Average |
| B) <u>– 1</u> | B) 1st | B) –1 | B)  | B) Clear one (–1) | B) Higher |
| 2)<br>A) 10** | | | A)  | A) Press ten (10) | A) Average |

*Verbal Count: The printed number indicates the number to be spoken. The number in the parenthesis indicates the value added or subtracted with the manipulation.

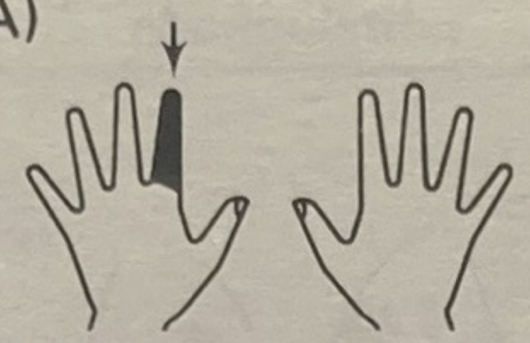
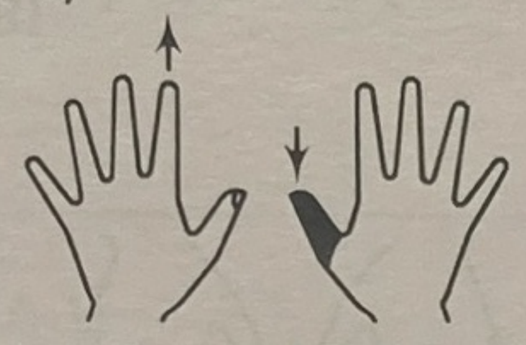
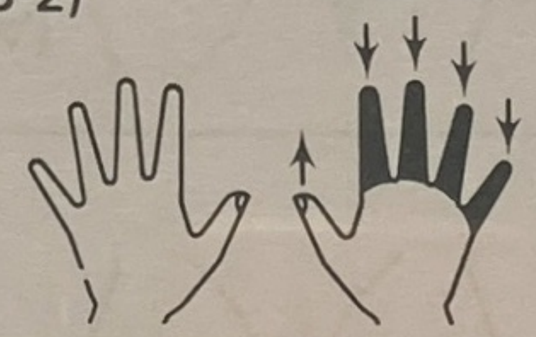
**For the purpose of concentrating on key manipulations of Subtraction Stage 2, 10 is pressed here directly. It could represent the total of several addends.

| Example | Stage | Breakdown of No. | Manipulation | Verbal Count* | Voice Intonation |
|---|---|---|---|---|---|
| B) - 1 | B) 1st | B) -1 | B)  | B) Clear one (-1) | B) Higher |
| 3)<br>A) 9 | A-1) 2nd | A-1) 5 | A-1)  | A-1) Press five (5) | A-1) Average |
| | A-2) 1st | A-2) 1 + 1 + 1 + 1 | A-2)  | A-2) six (1), seven (1), eight (1), nine (1) | A-2) Average |

| Example | Stage | Breakdown of No. | Manipulation | Verbal Count* | Voice Intonation |
|---|---|---|---|---|---|
| B) - 5 | B) 2nd | B) -5 | B)  | B) Clear five (-5) | B) Average |
| 4)<br>A) 14** | | | A)  | A) Press fourteen (14) | A) Average |
| B) - 5 | B) 2nd | B) -5 | B)  | B) Interchange five (-10+5) | B) Average |

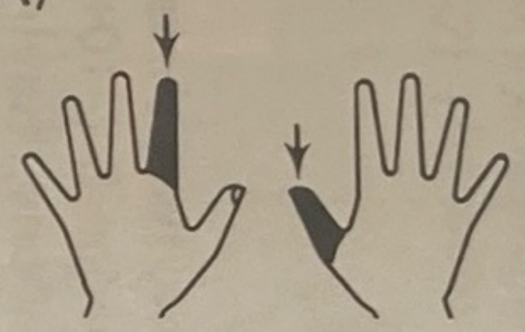
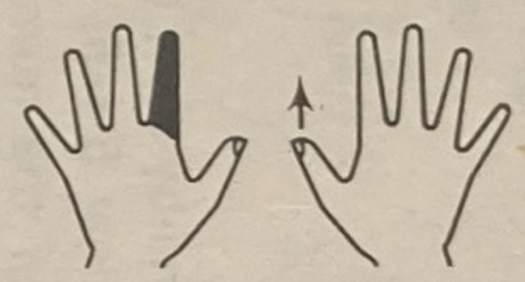
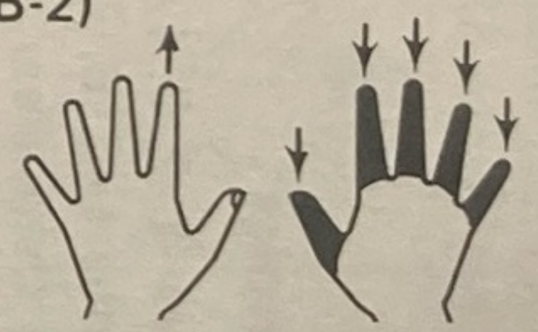
*Verbal Count: The printed number indicates the number to be spoken. The number in the parenthesis indicates the value added or subtracted with the manipulation.

**For the purpose of concentrating on key manipulations of Subtraction Stage 2, 14 is pressed here directly. It could represent the total of several addends.

| Example | Stage | Breakdown of No. | Manipulation | Verbal Count* | Voice Intonation |
|---|---|---|---|---|---|
| 5)<br>A) 10** | | | A)  | A) Press ten (10) | A) Average |
| B) - 6 | B-1) 2nd | B-1) -5 | B-1)  | B-1) Interchange five (–10+5) | B-1) Average |
| | B-2) 1st | B-2) -1 | B-2)  | B-2) six (–1) | B-2) Higher |

*Verbal Count: The printed number indicates the number to be spoken. The number in the parenthesis indicates the value added or subtracted with the manipulation.

**For the purpose of concentrating on key manipulations of Subtraction Stage 2, 10 is pressed here directly. It could represent the total of several addends.

| Example | Stage | Breakdown of No. | Manipulation | Verbal Count* | Voice Intonation |
|---|---|---|---|---|---|
| 6)<br>A) 15** | | | A)  | A) Press fifteen (15) | A) Average |
| B) - 6 | B-1) 2nd | B-1) –5 | B-1)  | B-1) Clear five (–5) | B-1) Average |
| | B-2) 1st | B-2) –1 | B-2)  | B-2) one (–1) | B-2) Higher |

*Verbal Count: The printed number indicates the number to be spoken. The number in the parenthesis indicates the value added or subtracted with the manipulation.

**For the purpose of concentrating on key manipulations of Subtraction Stage 2, 15 is pressed here directly. It could represent the total of several addends.

Subtraction Stage 2 examples requiring the subtraction of numbers 5 through 9 are arranged in two sets.

In the first set, five is always pressed and available to be cleared directly. In the second set, five is not pressed and has to be subtracted by Interchange 5 (-10+5). When the student has mastered the six key Stage 2 Subtraction manipulations described in the preceding pages, the following exercise drills should be performed daily.

1) Press 50 and clear 5's in sequence saying "Big Interchange 5 (-50+40+5), clear five (-5), Interchange five (-10+5), clear five (-5), Interchange five (-10+5), etc." until zero is reached.

2) Press 60 and clear 6's in sequence saying "Interchange five (-10+5) and six (-1), Big Interchange five (-50+40+5) and six (-1), clear five (-5) and six (-1), Interchange five (-10+5) and six (-1), etc." Until zero is reached.

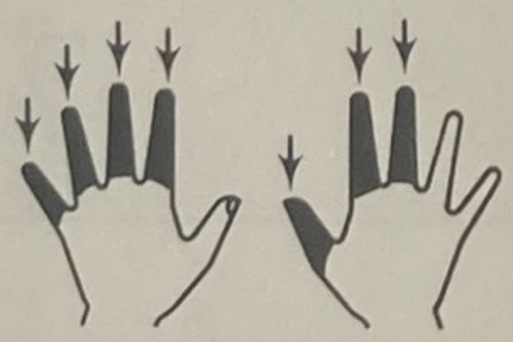
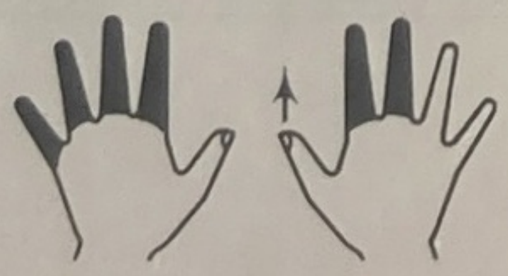
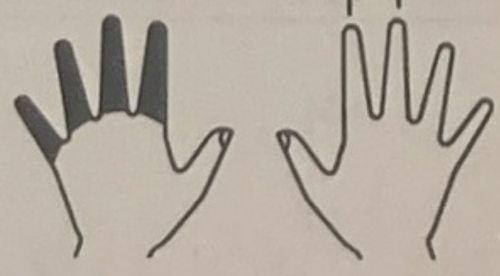
3) Press 70 and clear 7's in sequence until zero is reached.

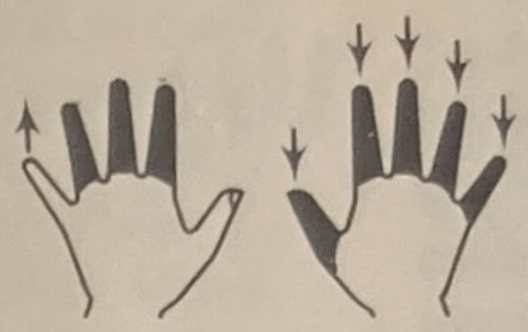
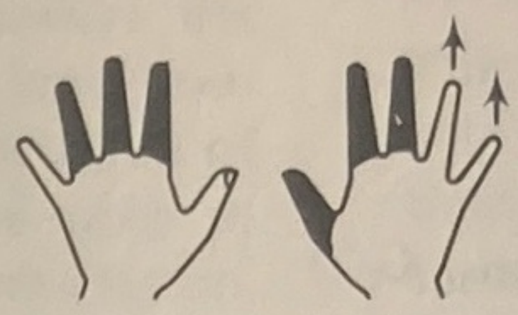
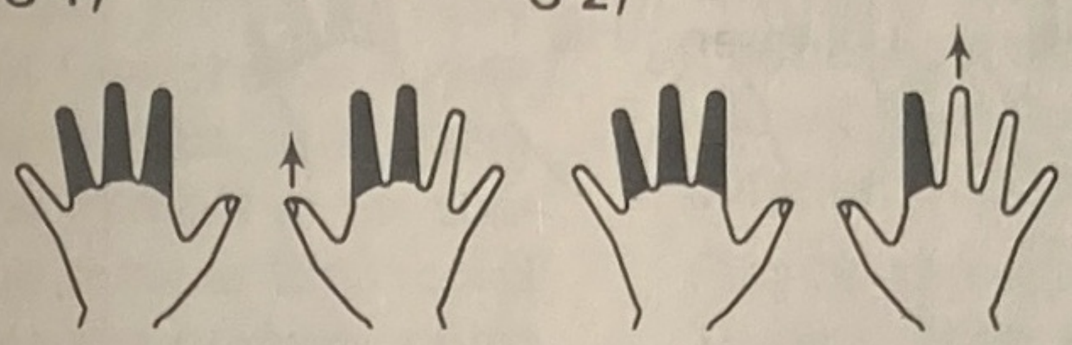
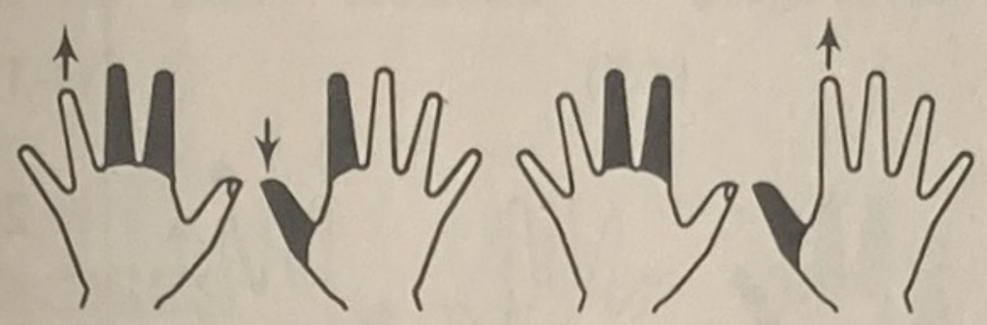
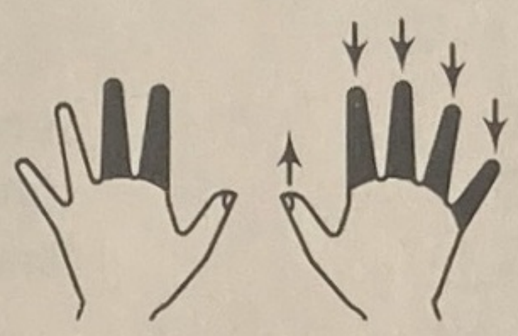
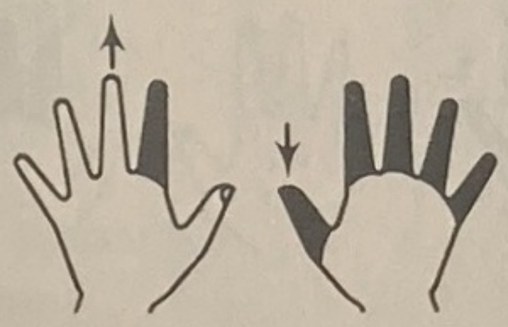
4) Press 80 and clear 8's in sequence until zero is reached.

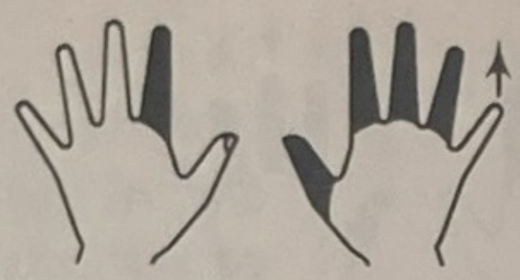
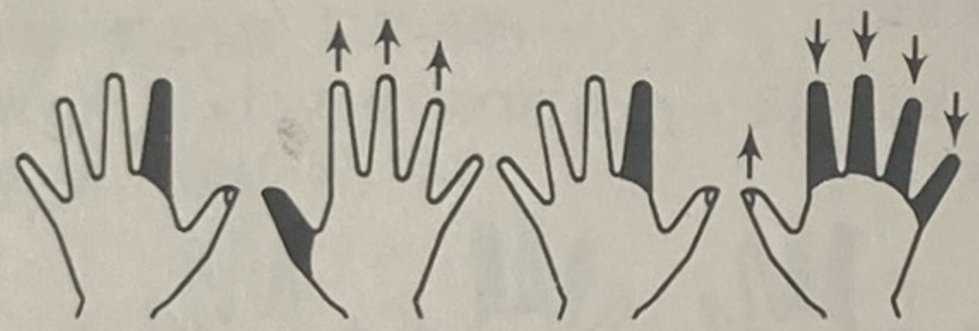
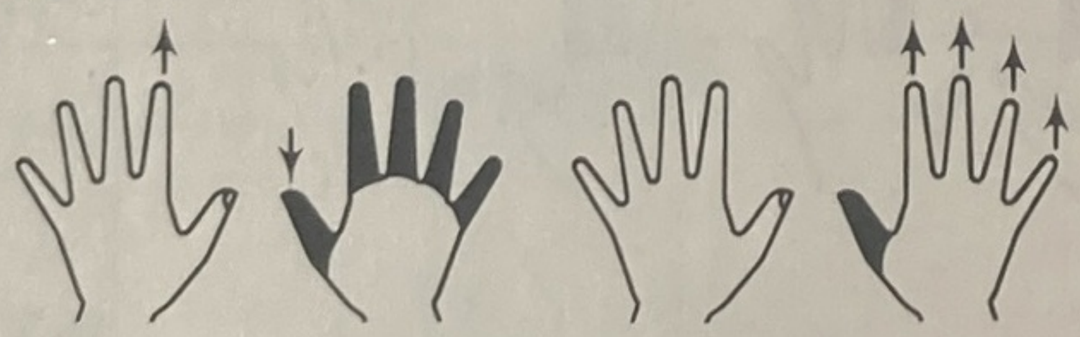
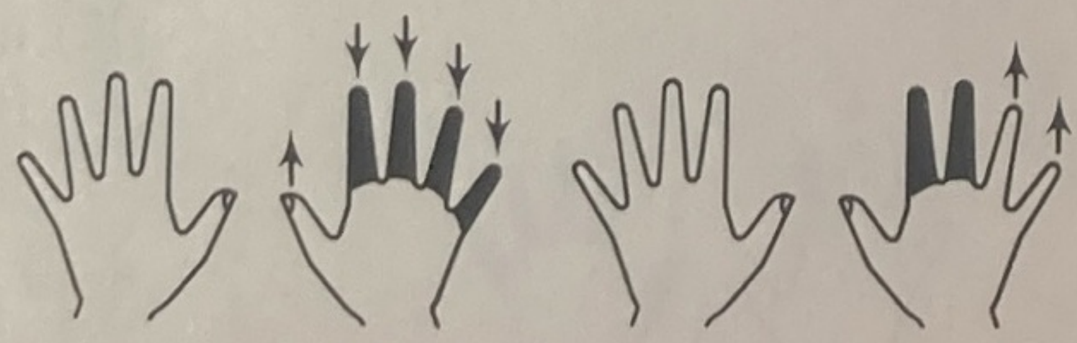
5) Press 90 and clear 9's in sequence until zero is reached.

## Manipulation Examples

a) First Stage Numbers—1, 2, 3, 4,—clear, one-by-one, (saying: one, two, three, four).
b) Second Stage Numbers—5, 6, 7, 8, 9,—clear 5 (-5) or Interchange 5 (-10+5) directly (saying: five) and then clear the rest of the number, one-by-one, (saying six (-1), seven (-1), eight (-1), nine (-1).)

| Number | Stage | Manipulation | Verbal Count |
|---|---|---|---|
| 47 | Already Pressed |  | Press Left Hand: onety (10), twenty (10), thirty (10), forty (10). Right Hand: five (5), six (1), seven (1). |
| -8 | A-1) 2nd<br>A-2) 1st | A-1)  A-2)  | A-1) clear five (-5)<br>A-2) six (-1), seven (-1) |

| Number | Stage | Manipulation | Verbal Count |
|---|---|---|---|
| | A-3) 1st | A-3)  | A-3) eight (–1) |
| –2 | B) 1st | B)  | B) clear one (–1), two (–1) |
| –6 | C-1) 2nd<br>C-2) 1st | C-1) C-2)  | C-1) clear five (–5)<br>C-2) six (–1) |
| –7 | D-1) 2nd<br>D-2) 1st | D-1) D-2)  | D-1) Interchange 5 (–10+5)<br>D-2) six (–1) |
| | D-3) 1st | D-3)  | D-3) seven (–1) |
| –5 | E) 2nd | E)  | E) Interchange 5 (–10+5) |

| Number | Stage | Manipulation | Verbal Count |
|---|---|---|---|
| -1 | F) 1st | F)  | F) clear one (-1) |
| -4 | G-1) 1st<br>G-2) 1st | G-1) G-2)  | G-1) clear one (-1), two (-1), three (-1)<br>G-2) four (-1) |
| -9 | H-1) 2nd<br>H-2) 1st | H-1) H-2)  | H-1) Interchange 5 (-10+5)<br>H-2) six (-1), seven (-1), eight (-1), nine (-1) |
| -3<br>2 | I-1) 1st<br>I-2) 1st | I-1) I-2)  | I-1) clear one (-1)<br>I-2) two (-1), three (-1) |

## STAGE 2

In the following series of exercises in Subtraction Stage 2, all addends should be pressed at the highest level the students know, i.e., Stage 2, Stage 3, or Stage 4. The first section contains problems where the subtrahends belong to Group 1, i.e. they must be cleared, *one-by-one.* Moreover, the number cleared is always the same as a number that has just been added. By reversing the manipulation of addition, the students will be able to concentrate on the proper manipulation.

In the problems with Group 2 subtrahends, five must be cleared, either directly or indirectly using an Interchange 5, and then the rest of the number is cleared, one-by-one. Again the number in the subtrahend is exactly the same as the second addend, thereby giving the students a hint as to how to operate, i.e., reverse the previous manipulation. In both sections instructions are provided at the end of the row.

The next section provides examples using Group 2 subtrahends. The instructions are no longer found in each row but are coded with the symbols • and ⊙ . Moreover, the subtraction manipulations are no longer the reverse manipulations from the previous addend.

Then there are examples with Group 1 subtrahends, but the instructions are not marked nor are the numbers in the subtrahend linked with the number in the minuend or the addends.

This is followed by examples with Group 2 subtrahends without giving instructions and without linking the subtraction manipulation to prior manipulations.

Finally there are examples using Power Trading. Whenever a clearing is called for, the clearing manipulation should be a Stage 2 manipulation.

## PRACTICE EXERCISES

Group I

Clear one-by-one saying, one, two, three, four.

```
  1    2 ←→(Press, one-by-one)
+ 1  + 1 ←→(Press, one-by-one)
- 1  - 1 ←→(Clear, one-by-one)
```

```
  3    4 ←→(Press, one-by-one)
+ 1  + 1 ←→(Press, one-by-one)
- 1  - 1 ←→(Clear, one-by-one)
```

```
  5    6 ←→(5 Available for pressing)
+ 1  + 1 ←→(Press, one-by-one)
- 1  - 1 ←→(Clear, one-by-one)
```

```
  7    8 ←→(5 Available for pressing)
+ 1  + 1 ←→(Press, one-by-one)
- 1  - 1 ←→(Clear, one-by-one)
```

```
  1    2 ←→(Press, one-by-one)
+ 2  + 2 ←→(Press, one-by-one)
- 2  - 2 ←→(Clear, one-by-one)
```

```
  3    4 ←→(Press, one-by-one)
+ 2  + 2 ←→(Press, one-by-one)
- 2  - 2 ←→(Clear, one-by-one)
```

```
  5    6 ←→(5 Available for pressing)
+ 2  + 2 ←→(Press, one-by-one)
- 2  - 2 ←→(Clear, one-by-one)
```

```
   7     8 ←→ (5 Available for pressing)
+  2  +  2 ←→ (Press, one-by-one)
-  2  -  2 ←→ (Clear, one-by-one)
```

```
   1     2 ←→ (Press, one-by-one)
+  3  +  3 ←→ (Press, one-by-one)
-  3  -  3 ←→ (Clear, one-by-one)
```

```
   3     4 ←→ (Press, one-by-one)
+  3  +  3 ←→ (Press, one-by-one)
-  3  -  3 ←→ (Clear, one-by-one)
```

```
   5     6 ←→ (5 Available for pressing)
+  3  +  3 ←→ (Press, one-by-one)
-  3  -  3 ←→ (Clear, one-by-one)
```

```
   8     9 ←→ (5 Available for pressing)
+  3  +  3 ←→ (Press, one-by-one)
-  3  -  3 ←→ (Clear, one-by-one)
```

```
   1     2 ←→ (Press, one-by-one)
+  4  +  4 ←→ (Press, one-by-one)
-  4  -  4 ←→ (Clear, one-by-one)
```

```
   3     4 ←→ (Press, one-by-one)
+  4  +  4 ←→ (Press, one-by-one)
-  4  -  4 ←→ (Clear, one-by-one)
```

```
   5     6 ←→ (5 Available for pressing)
+  4  +  4 ←→ (Press, one-by-one)
-  4  -  4 ←→ (Clear, one-by-one)
```

```
   8     9 ←→ (5 Available for pressing)
+  4  +  4 ←→ (Press, one-by-one)
-  4  -  4 ←→ (Clear, one-by-one)
```

Group 2

Clear 5 directly or Interchange 5 (-10+5). Then clear one-by-one saying, "Six, seven, eight, nine."

```
←→ (Press, one-by-one)
←→ (5 Available for pressing)
←→ (5 Available for clearing)
```

```
   1     2     3     4     1
+  5  +  5  +  5  +  5  +  6
-  5  -  5  -  5  -  5  -  6
```

```
←→ (5 Available for pressing)
←→ (5 Not Available for pressing)
←→ (5 Not Available for clearing)
```

```
   5     6     7     8     9
+  5  +  5  +  5  +  5  +  5
-  5  -  5  -  5  -  5  -  5
```

```
←→ (Press, one-by-one)
←→ (5 Available for pressing)
←→ (5 Available for clearing)
```

```
   1     2     1     2     3
+  7  +  7  +  8  +  6  +  6
-  7  -  7  -  8  -  6  -  6
```

```
←→ (5 Available for pressing)
←→ (5 Not Available for pressing)
←→ (5 Not Available for clearing)
```

```
   5     5     6     7     8
+  6  +  9  +  6  +  6  +  6
-  6  -  9  -  6  -  6  -  6
```

←→(5 *Available* for pressing)
←→(5 *Not Available* for pressing)
←→(5 *Not Available* for clearing)

| | | | | |
|---|---|---|---|---|
| 5 | 5 | 6 | 6 | 7 |
| + 7 | + 8 | + 7 | + 8 | + 7 |
| − 7 | − 8 | − 7 | − 8 | − 7 |

←→(Press, one-by-one)
←→(5 *Available* for pressing)
←→(5 *Not Available* for clearing)

| | | | | |
|---|---|---|---|---|
| 2 | 3 | 4 | 4 | 4 |
| + 8 | + 8 | + 6 | + 7 | + 8 |
| − 8 | − 8 | − 6 | − 7 | − 8 |

←→(5 *Available* for pressing)
←→(5 *Not Available* for pressing)
←→(5 *Available* for clearing)

| | | | | |
|---|---|---|---|---|
| 7 | 8 | 9 | 9 | 9 |
| + 8 | + 8 | + 6 | + 7 | + 8 |
| − 8 | − 8 | − 6 | − 7 | − 8 |

←→(Press, one-by-one)
←→(5 *Available* for pressing)
←→(5 *Not Available* for clearing)

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 3 | 4 |
| + 9 | + 9 | + 7 | + 9 | + 9 |
| − 9 | − 9 | − 7 | − 9 | − 9 |

←→(5 *Available* for pressing)
←→(5 *Not Available* for pressing)
←→(5 *Available* for clearing)

| | | | | |
|---|---|---|---|---|
| 6 | 7 | 8 | 8 | 9 |
| + 9 | + 9 | + 7 | + 9 | + 9 |
| − 9 | − 9 | − 7 | − 9 | − 9 |

Group 2
(without regrouping)

Fifth finger is available to clear. When number 5 is marked "• " say, "Clear five."

| | | | |
|---|---|---|---|
| 4 | 5 | 3 | 9 |
| 1 | 2 | 3 | 3 |
| 2 | 1 | 2 | 6 |
| − • 5 | − • 5 | − • 5 | − • 5 |

Fifth finger is not available to clear. When number 5 is marked "⊙," say "Interchange five."

| | | | |
|---|---|---|---|
| 5 | 3 | 7 | 4 |
| 2 | 2 | 1 | 1 |
| 3 | 6 | 4 | 8 |
| − ⊙ 5 | − ⊙ 5 | − ⊙ 5 | − ⊙ 5 |

Clear fifth finger and 1 more when 6 is marked "• ." Say "Clear five-six."

| | | | |
|---|---|---|---|
| 2 | 5 | 3 | 9 |
| 4 | 1 | 3 | 9 |
| 1 | 2 | 3 | 1 |
| − • 6 | − • 6 | − • 6 | − • 6 |

Interchange 5 (−10+5) and 1 more when 6 is marked "⊙ ." Say "Interchange five-six."

| | | | |
|---|---|---|---|
| 3 | 6 | 4 | 6 |
| 3 | 1 | 2 | 3 |
| 4 | 4 | 6 | 4 |
| − ⊙ 6 | − ⊙ 6 | − ⊙ 6 | − ⊙ 6 |

Clear fifth finger and 2 more when 7 is marked "• ." Say "Clear five-six-seven."

| | | | |
|---|---|---|---|
| 1 | 5 | 6 | 8 |
| 2 | 2 | 1 | 1 |
| 5 | 8 | 9 | 8 |
| − • 7 | − • 7 | − • 7 | − • 7 |

Interchange 5 (–10+5) and 2 more when 7 is marked "⊙ ." Say "Interchange five-six-seven."

| | | | |
|---|---|---|---|
| 3 | 2 | 3 | 7 |
| 1 | 4 | 5 | 5 |
| 6 | 5 | 4 | 1 |
| – ⊙ 7 | – ⊙ 7 | – ⊙ 7 | – ⊙ 7 |

Clear fifth finger and 3 more when 8 is marked "• ." Say "Clear five-six-seven-eight."

| | | | |
|---|---|---|---|
| 8 | 4 | 5 | 1 |
| 6 | 4 | 2 | 8 |
| 2 | 9 | 8 | 9 |
| – • 8 | – • 8 | – • 8 | – • 8 |

Interchange 5 (–10+5) and 3 more when 8 is marked "⊙ ." Say "Interchange five-six-seven-eight."

| | | | |
|---|---|---|---|
| 4 | 2 | 4 | 6 |
| 4 | 8 | 3 | 5 |
| 2 | 1 | 5 | 2 |
| – ⊙ 8 | – ⊙ 8 | – ⊙ 8 | – ⊙ 8 |

Clear fifth finger and 4 more when 9 is marked "• ." Say "Clear five-six-seven-eight-nine."

| | | | |
|---|---|---|---|
| 7 | 4 | 2 | 1 |
| 6 | 4 | 8 | 8 |
| 3 | 7 | 6 | 9 |
| – • 9 | – • 9 | – • 9 | – • 9 |

Interchange 5 (–10+5) and 4 more when 9 is marked "⊙ ." Say "Interchange five-six-seven-eight-nine."

| | | | |
|---|---|---|---|
| 3 | 9 | 6 | 4 |
| 2 | 1 | 3 | 5 |
| 5 | 1 | 3 | 4 |
| – ⊙ 9 | – ⊙ 9 | – ⊙ 9 | – ⊙ 9 |

Group 1

Clear one-by-one saying, "One, two, three, four."

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 2 | 3 |
| – 1 | – 1 | – 1 | – 1 | – 2 | – 2 |

| | | | | | |
|---|---|---|---|---|---|
| 7 | 8 | 9 | 7 | 8 | 9 |
| – 1 | – 1 | – 1 | – 2 | – 2 | – 2 |

| | | | | | |
|---|---|---|---|---|---|
| 5 | 5 | 5 | 5 | 6 | 6 |
| – 1 | – 2 | – 3 | – 4 | – 2 | – 3 |

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 2 | 3 | 3 | 3 |
| 9 | 8 | 9 | 7 | 8 | 9 |
| – 1 | – 2 | – 2 | – 3 | – 3 | – 3 |

| | | | | | |
|---|---|---|---|---|---|
| 9 | 8 | 7 | 6 | 9 | 8 |
| 1 | 2 | 3 | 4 | 2 | 3 |
| – 1 | – 2 | – 3 | – 4 | – 2 | – 3 |

| | | | | | |
|---|---|---|---|---|---|
| 8 | 8 | 9 | 9 | 8 | 8 |
| 2 | 2 | 1 | 1 | 2 | 3 |
| – 1 | – 3 | – 2 | – 3 | – 4 | – 2 |

| | | | | | |
|---|---|---|---|---|---|
| 5 | 5 | 5 | 5 | 5 | 5 |
| 5 | 6 | 7 | 8 | 5 | 6 |
| – 1 | – 2 | – 3 | – 4 | – 2 | – 3 |

Group 2

Clear 5 directly or Interchange 5 (–10+5). Then clear one-by-one saying, "Six, seven, eight, nine."

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | 7 | 8 | 9 | 7 | 8 | 9 |
| – 5 | – 5 | – 5 | – 5 | – 6 | – 6 | – 6 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 | 5 | 5 | 5 | 5 | 6 | 6 |
| 5 | 6 | 7 | 8 | 9 | 4 | 5 |
| – 5 | – 5 | – 5 | – 5 | – 5 | – 6 | – 6 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | 6 | 7 | 7 | 7 | 7 | 7 |
| 8 | 9 | 3 | 4 | 5 | 6 | 7 |
| – 6 | – 6 | – 7 | – 7 | – 7 | – 7 | – 7 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| – 8 | – 8 | – 8 | – 8 | – 8 | – 8 | – 8 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 9 | 9 | 9 | 9 | 9 | 9 | 9 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| – 9 | – 9 | – 9 | – 9 | – 9 | – 9 | – 9 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| 9 | 8 | 9 | 7 | 8 | 9 | 6 |
| – 5 | – 6 | – 7 | – 8 | – 9 | – 5 | – 6 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 1 | 2 | 2 | 1 |
| 9 | 8 | 9 | 9 | 8 | 9 | 9 |
| – 6 | – 7 | – 8 | – 7 | – 8 | – 9 | – 8 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | 3 | 3 | 3 | 4 | 4 | 4 |
| 8 | 7 | 8 | 9 | 6 | 7 | 8 |
| – 5 | – 6 | – 7 | – 8 | – 9 | – 5 | – 6 |

## HYPAI POWER TRADING

### Calculate Column-by-Column after HYPAI Power Trades

| □ | □ | □ | □ | | | |
|---|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) | | |
| | 8, | 0 | 0 | 7 | | |
| | – | | | 9 | 6 | |

| □ | □ | □ | □ | | | |
|---|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | | |
| | 5, | 0 | 0 | 1 | | |
| | – | | | 8 | 0 | |

| □ | □ | □ | □ | | | |
|---|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | | |
| | 9, | 0 | 0 | 4 | | |
| | – | | | 3 | 1 | |

| □ | □ | □ | □ | | | |
|---|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | | |
| | 4, | 0 | 7 | 3 | | |
| | – | | 6 | 7 | 9 | |

| □ | □ | □ | □ | | | |
|---|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) | | |
| | 6, | 0 | 4 | 0 | | |
| | – | | 9 | 4 | 3 | |

| □ | □ | □ | □ | | | |
|---|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | | |
| | 7, | 0 | 8 | 2 | | |
| | – | | 5 | 8 | 6 | |

| □ | □ | □ | □ | | | |
|---|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) | | |
| | 1, | 0 | 0 | 0 | | |
| | – | | | | 1 | |

| □ | □ | □ | □ | | | |
|---|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | | |
| | 2, | 0 | 0 | 0 | | |
| | – | | | | 5 | |

| □ | □ | □ | □ | | | |
|---|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | | |
| | 3, | 0 | 0 | 0 | | |
| | – | | | | 8 | |

| □ | □ | □ | □ | | | |
|---|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | | |
| | 6, | 7 | 9 | 3 | | |
| | –3, | 2 | 4 | 8 | | |

| □ | □ | □ | □ | | | |
|---|---|---|---|---|---|---|
| (–1) | (9) | (9) | (9) | (10) | | |
| | 6, | 7 | 2 | 9 | | |
| | –4, | 9 | 5 | 2 | | |

| □ | □ | □ | □ | | | |
|---|---|---|---|---|---|---|
| ( ) | ( ) | ( ) | ( ) | ( ) | | |
| | 5, | 0 | 4 | 1 | | |
| | –1, | 8 | 3 | 7 | | |

□ □ □ □ □ □

(–2) (18) (18) (18) (20)

9, 0 2 4

– 8 7 6

– 5 3

□ □ □ □ □ □

( ) ( ) ( ) ( ) ( )

– 5 3

2 0 0

– 1

7, 5 1 8

□ □ □ □ □ □

( ) ( ) ( ) ( ) ( )

4, 1 6 0

– 7 9

– 3 5 7

□ □ □ □ □ □

(–2) (18) (18) (18) (20)

3, 2 5 6

– 9 2 8

4, 0 1 2

– 3 7

□ □ □ □ □ □

(–2) (18) (18) (18) (20)

– 9

6, 2 3 0

– 7 9

7 8 1

□ □ □ □ □ □

( ) ( ) ( ) ( ) ( )

5, 2 7 5

6, 3 1 0

– 8 9

– 5 4 5

□ □ □ □ □ □

( ) ( ) ( ) ( ) ( )

7, 0 6 3

– 1 8 9

– 3 4

□ □ □ □ □ □

( ) ( ) ( ) ( ) ( )

3 4 2

– 9 6

2, 0 0 1

– 4

□ □ □ □ □ □

( ) ( ) ( ) ( ) ( )

8, 2 0 4

– 8 6

– 9 7 3

□ □ □ □ □ □

( ) ( ) ( ) ( ) ( )

– 7 0 6

– 5 8

6, 0 1 5

9, 2 4 3

□ □ □ □ □ □

( ) ( ) ( ) ( ) ( )

7, 4 0 7

– 4

– 9 8

2 6 0

□ □ □ □ □ □

( ) ( ) ( ) ( ) ( )

– 7 2

8, 4 0 6

– 9 5 3

7, 1 8 9

## SUBTRACTION STAGE 3

As in the earlier stages, Subtraction Stage 3 is the exact reverse of Addition Stage 3. Thus to perform 3rd stage subtraction, any number between 1 and 9 must be cleared in one single simultaneous manipulation. The method of clearing any single digit number obviously depends upon what fingers are pressed on the right hand before the clearing operation is begun. The following pages will illustrate the three different ways that the number 3 can be cleared. This will be followed by illustrations of how the number 8 can be cleared. Finally, a pattern chart showing how to clear all the numbers 1-9 is displayed.

After studying the pattern chart very carefully, the student can begin to practice Stage 3 with the exercises that follow the chart.

As in every other stage, Subtraction Stage 3 can not begin unless Addition Stage 3 is fully understood. It is better to delay the introduction of the subtraction method until the students are completely comfortable with the Addition Stage 3 method.

### Points to Keep in Mind

Remember that in the beginning of 3rd stage subtraction students require considerable practice with the teacher giving oral clues describing the manipulation. As the students perform each manipulation they should repeat the clues the teacher has given.

**Example A: (Subtracting the number 3)**

There are three different ways of subtracting 3. In all cases, the fingers already pressed on the right hand determine the pattern to be used. To subtract 3, use one of the following patterns.

*Case 1: (–3)*

When three fingers on the right hand are available for clearing, then clear them. *When three right-hand fingers are not available for clearing use either:*

*Case 2: (–5+2)*

The three right-hand fingers are not available for clearing, but the fifth finger is available for clearing.

*Case 3: (–10+7)*

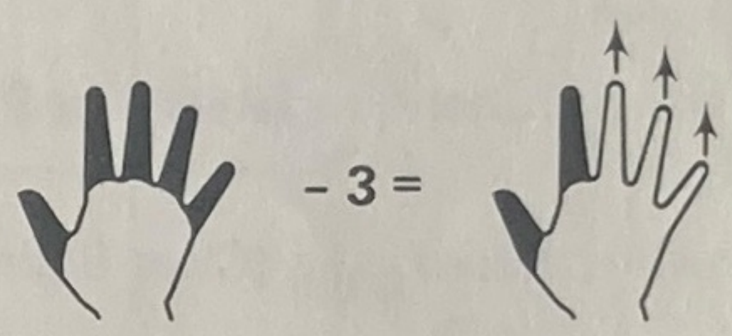
The three right-hand fingers and the fifth finger are not available for clearing.

Note: When clearing fingers, you must always work toward the thumb.

Illustration Example A: *Case 1: (–3)*

Three right-hand fingers are available for clearing. When 3 is subtracted from the numbers 3, 4, 8, or 9 (already pressed on the right hand), we must use the pattern that employs the three fingers available for clearing.

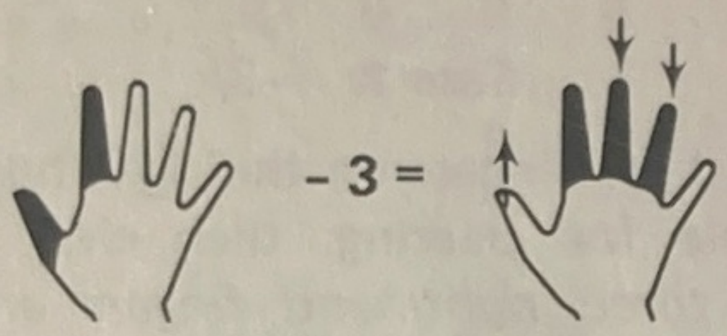
*9 is already pressed* (Clear 3 directly)




*Case 2: (–5+2)*

When we subtract 3 from 5, 6, or 7 (already pressed on the right hand), we must use the (–5+2)—clear 5, press 2-manipulation.

*6 is already pressed* (Clear 5, Press 2 simultaneously)




*Case 3: (–10+7)*

When the right hand registers 0, 1, or 2, we must use the pattern that employs the (–10+7)—clear 10, press 7—manipulation.

*12 is already pressed* (Clear 10, Press 7 simultaneously)

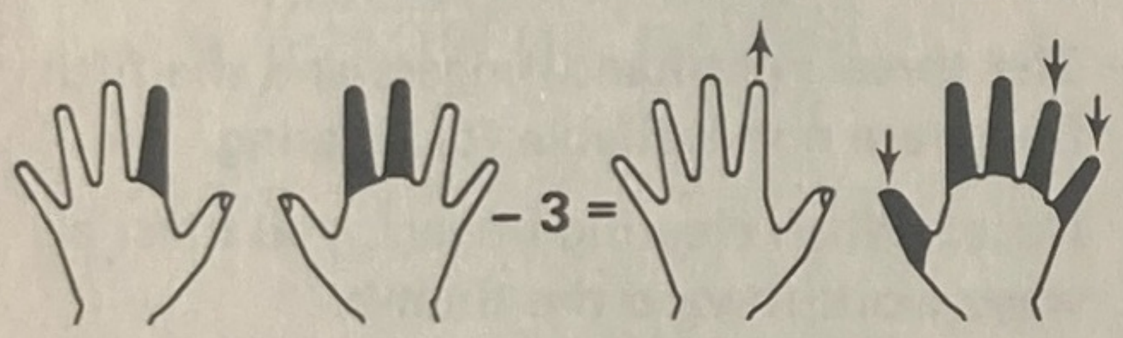


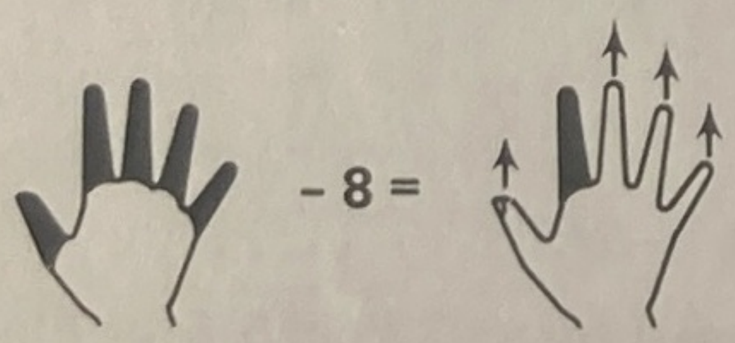

Illustration Example B:
(Subtracting the number 8)

There are three different ways of subtracting 8. Again it depends on which fingers are pressed on the right hand. To clear 8, use one of the following patterns.

*Case 1: (–8)*

When the right hand registers 8 or 9, clear directly.

*9 is already pressed* (Clear 8 directly)

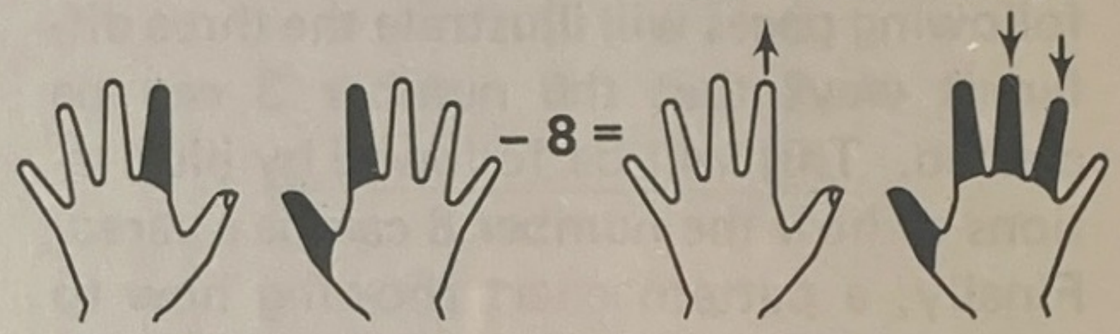



*When the number 8 or 9 is not pressed on the right hand, 8 is subtracted by using either:*

*Case 2: (–10+2)*

When the right hand registers 0, 1, 2, 5, 6, or 7, we must use the (–10+2)—clear 10, press 2—manipulation.

*16 is already pressed* (Clear 10, Press 2 simultaneously)



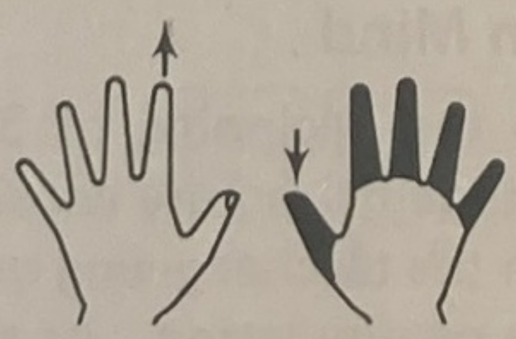

*Case 3: (Interchange 5–3)*

When the right hand registers 3 or 4, we must use the Interchange 5–3 manipulation.

*14 is already pressed*

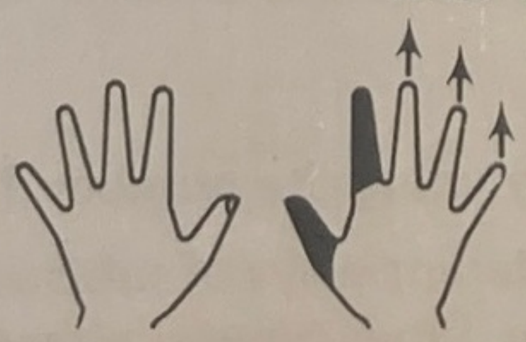
Interchange 5 (–10+5)
Through clear 10, press 5 manipulation

A.



and simultaneously clear 3

B.



## SUBTRACTION STAGE 3 PATTERN CHART

Subtraction Stage 3 manipulations are combinations of the following mathematical patterns employed in subtracting certain numbers. The fingers still pressed on the right hand determine which pattern to use for subtraction. The key concept of Subtraction Stage 3 is to clear all fingers in a given manipulation simultaneously.

| Pattern / Subtrahends | Pattern 1 | Pattern 2 | Pattern 3 | Pattern 4 | Pattern 5 |
|---|---|---|---|---|---|
| -1 | -1 Available for Clearing | | -5+4 | -10+9 (Interchange 5+4)* | |
| -2 | -2 Available for Clearing | | -5+3 | -10+8 (Interchange 5+3)* | |
| -3 | -3 Available for Clearing | | -5+2 | -10+7 (Interchange 5+2)* | |
| -4 | -4 Available for Clearing | | -5+1 | -10+6 (Interchange 5+1)* | |
| -5 | -5 Available for Clearing | Interchange 5 (-10+5) | | | |
| -6 | -6 Available for Clearing | | | -10+4 | Interchange 5–1 |
| -7 | -7 Available for Clearing | | | -10+3 | Interchange 5–2 |
| -8 | -8 Available for Clearing | | | -10+2 | Interchange 5–3 |
| -9 | -9 Available for Clearing | | | -10+1 | Interchange 5–4 |

*The instructions in the parentheses can be used as a temporary manipulation until the student gets used to the more sophisticated maneuver as instructed above the parentheses.

## EXERCISES

Pattern 1 (Clear Directly)

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | 3 | 3 | 4 | 4 | 6 | 6 |
| - 1 | - 1 | - 2 | - 1 | - 3 | - 1 | - 5 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 7 | 8 | 8 | 9 | 9 | 4 | 7 |
| - 6 | - 1 | - 7 | - 1 | - 8 | - 2 | - 5 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 8 | 9 | 9 | 8 | 9 | 9 |
| - 2 | - 6 | - 2 | - 7 | - 3 | - 3 | - 6 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 9 | 0 | 1 | 2 | 3 | 4 |
| - 5 | - 5 | - 0 | - 1 | - 2 | - 3 | - 4 |

Pattern 2 (Interchange 5 (Clear 10, Press 5))

| | | | | |
|---|---|---|---|---|
| 5 | 6 | 7 | 8 | 9 |
| 5 | 5 | 5 | 5 | 5 |
| - 5 | - 5 | - 5 | - 5 | - 5 |

| | | | | |
|---|---|---|---|---|
| 5 | 5 | 5 | 5 | 5 |
| 5 | 6 | 7 | 8 | 9 |
| - 5 | - 5 | - 5 | - 5 | - 5 |

Pattern 3 (Clear 5, Press ___)

| | | | | |
|---|---|---|---|---|
| 5 | 6 | 7 | 8 | 5 |
| - 4 | - 4 | - 4 | - 4 | - 3 |

| | | | | |
|---|---|---|---|---|
| 6 | 7 | 5 | 6 | 5 |
| - 3 | - 3 | - 2 | - 2 | - 1 |

Pattern 4 (Clear 10, Press ___)

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 6 | 7 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 |
| - 9 | - 9 | - 9 | - 9 | - 9 | - 9 | - 9 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 9 | 9 | 9 | 9 | 9 | 9 | 9 |
| 1 | 2 | 3 | 4 | 6 | 7 | 8 |
| - 9 | - 9 | - 9 | - 9 | - 9 | - 9 | - 9 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | 3 | 4 | 7 | 8 | 9 | 3 |
| 8 | 8 | 8 | 8 | 8 | 8 | 7 |
| - 8 | - 8 | - 8 | - 8 | - 8 | - 8 | - 7 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 8 | 8 | 8 | 8 | 8 | |
| 2 | 3 | 4 | 7 | 8 | 9 | |
| - 8 | - 8 | - 8 | - 8 | - 8 | - 8 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 9 | 4 | 9 | 6 | 7 | |
| 7 | 7 | 6 | 6 | 4 | 4 | |
| - 7 | - 7 | - 6 | - 6 | - 4 | - 4 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| 7 | 7 | 6 | 6 | 4 | 4 | |
| 8 | 9 | 4 | 9 | 6 | 7 | |
| - 7 | - 7 | - 6 | - 6 | - 4 | - 4 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 9 | 7 | 8 | 9 | 8 | |
| 4 | 4 | 3 | 3 | 3 | 2 | |
| - 4 | - 4 | - 3 | - 3 | - 3 | - 2 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | 4 | 3 | 3 | 3 | 2 | |
| 8 | 9 | 7 | 8 | 9 | 8 | |
| - 4 | - 4 | - 3 | - 3 | - 3 | - 2 | |

Pattern 5 (Interchange 5, Clear ___)

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | 7 | 8 | 6 | 7 | 6 | 5 |
| 6 | 6 | 6 | 7 | 7 | 8 | 9 |
| - 6 | - 6 | - 6 | - 7 | - 7 | - 8 | - 9 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | 6 | 6 | 7 | 7 | 8 | 9 |
| 5 | 7 | 8 | 6 | 7 | 6 | 5 |
| - 6 | - 6 | - 6 | - 7 | - 7 | - 8 | - 9 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 2 | 4 | 5 | 6 |
| 2 | 1 | 3 | 5 | 5 | 4 | 7 |
| 8 | 9 | 5 | 6 | 4 | 5 | 1 |
| - 6 | - 6 | - 7 | - 7 | - 8 | - 8 | - 9 |

Combination problems of Pattern 1 and Pattern 2.

In sets A and B follow the instruction in the parentheses. In set C you must decide on the manipulation.

(A)

| | | | | |
|---|---|---|---|---|
| 2 | | | 4 | |
| - 1 | (Available) | | - 3 | (Available) |
| 8 | | | 6 | |
| - 6 | (Available) | | - 5 | (Available) |
| 5 | | | 6 | |
| - 3 | (Available) | | - 7 | (Available) |

```
  9
- 8 (Available)
  8
- 9 (Available)
  8
- 6 (Available)
```

```
(B) 8
  - 5 (Avail.)
  - 1 (Avail.)
    7
    9
  - 6 (Avail.)
```

```
  8 7 8 4 7 6 8
  6 6 1 9 7 9 5
  9 3 9 9 7 8 5
- 5-1-7-5-5-2-5
- 5-5-5-6-5-5-5
- 5-5-1-5-1-6-7
```

```
  7
- 2 (Avail.)
- 5 (Avail.)
  8
  6
- 3 (Avail.)
```

```
  9
- 6 (Avail.)
  2
  8
  9
- 7 (Avail.)
```

```
(C) 4 7 6 8 7 9
    9 8 2 1 2 8
  - 2-5-3-8-1-6
    6 9 9 7 1 7
  - 5-3-2-1-5-1
  - 2-5-2-2-3-6
```

Combination problems of Pattern 1 and Pattern 2.

```
  8
- 6 (Available)
  8
- 5 (Interchange 5)
  7
- 5 (Interchange 5)
```

```
  9
- 7 (Avail.)
  9
- 5 (Interch. 5)
  8
- 5 (Interch. 5)
```

```
  8
- 1 (Available)
  6
- 5 (Interchange 5)
  6
- 5 (Interchange 5)
```

```
  9
  4
- 5 (Interch. 5)
- 6 (Avail.)
  9
- 5 (Interch. 5)
```

```
  9
  3
- 5 (Interch. 5)
- 5 (Avail.)
  8
- 5 (Interch. 5)
```

```
  7
  4
- 5 (Interch. 5)
- 1 (Avail.)
  7
- 5 (Interch. 5)
```

Combination problems of Patterns 1, 2, and 3.

Get used to examining the fingers to see and to *feel* which pattern must be chosen.

```
  1
  7
- 4 (Clear 5, Press 1)
  8
- 5 (Interchange 5)
- 3 (Clear 5, Press 2)
```

```
  7
- 3 (Clear 5, Press 2)
  4
- 4 (Clear 5, Press 1)
  2
- 3 (Clear 5, Press 2)
```

```
  5
- 1 (Clear 5, Press 4)
  3
- 4 (Clear 5, Press 1)
  2
- 1 (Clear 5, Press 4)
```

```
  6 4 7 9 8 3 4
  7 9 8 5 2 9 8
  6 8 9 6 6 6 3
-8-5-3-5-4-1-5
-5-4-5-1-5-2-5
-2-2-3-3-3-1-2
```

Combination problems of Patterns 1, 2, 3, and 4.

The decision on which pattern to use rests with what fingers are pressed. Examine the fingers before making a decision.

```
  5
  9
- 3  (Available)
- 9  (Clear 10, Press 1)
  8
- 7  (Clear 10, Press 3)

  9
  8
- 7  (Available)
- 2  (Clear 10, Press 8)
  6
- 5  (Interchange 5)

  5
- 1  (Clear 5, Press 4)
  6
- 5  (Interch. 5)
  5
- 6  (Clear 10, Press 4)
```

```
  8   6   5   9   7   4   6
  5   8   9   7   8   6   6
- 4 - 5 - 2 - 8 - 1 - 5 - 3
- 7 - 6 - 3 - 4 - 3 - 1 - 7
  9   7   8   7   6   9   5
- 2 - 1 - 6 - 5 - 8 - 4 - 3
```

## HYPAI POWER TRADING

### Calculate Column-by-Column After HYPAI Power Trades

□ □ □ □ __
( ) ( ) ( ) ( ) ( )

```
  8, 7 5 3
-3, 0 6 9
  4, 2 5 1
-3, 5 9 4
```

□ □ □ □ __
(-2) (18) (18) (18) (20)

```
    - 6 9 4
  6, 0 4 2
    -   7 5
  9, 3 1 7
```

□ □ □ □ __
( ) ( ) ( ) ( ) ( )

```
    - 7 0 6
  5, 2 1 8
    -   4 3
  7, 3 6 9
```

□ □ □ □ __
( ) ( ) ( ) ( ) ( )

```
    - 1 2 9
  8, 4 3 0
    -   5 2
  4, 9 4 3
```

□ □ □ □ __
(-2) (18) (18) (18) (20)

```
  5, 0 0 0
    -     8
-7, 2 5 9
  8, 1 4 2
```

□ □ □ □ __
( ) ( ) ( ) ( ) ( )

```
  7, 0 0 0
    -     1
-8, 9 4 2
  6, 1 9 5
```

□ □ □ □ __
(-2) (18) (18) (18) (20)

```
  9, 2 5 4
    -   8 6
    -   9 5
      3 0 1
```

□ □ □ □ __
( ) ( ) ( ) ( ) ( )

```
  5, 7 3 0
    -   1 8
    -   6 4
      8 2 9
```

□ □ □ □ __
( ) ( ) ( ) ( ) ( )

```
  7, 2 5 3
-     9 0
-     7 4
    1 3 6
```

□ □ □ □ __
(-2) (18) (18) (18) (20)

```
      2 7 5
-1, 8 9 4
-3, 6 1 0
  8, 0 5 2
```

□ □ □ □ __
(-2) (18) (18) (18) (20)

```
  7, 5 9 4
-1, 8 6 3
  8, 7 5 0
-2, 5 4 2
```

□ □ □ □ __
( ) ( ) ( ) ( ) ( )

```
  8, 6 2 9
-2, 7 8 5
  9, 3 1 8
-4, 0 7 3
```

□ □ □ □ __
( ) ( ) ( ) ( ) ( )

```
    6 0 4
-2, 7 5 1
-1, 9 8 9
  9, 1 3 8
```

□ □ □ □ __
( ) ( ) ( ) ( ) ( )

```
    1 8 6
-1, 4 5 9
-3, 0 7 5
  7, 2 3 4
```

| □ □ □ □ | □ □ □ □ |
|---|---|
| (–2) (18) (18) (18) (20) | ( ) ( ) ( ) ( ) ( ) |
| 7, 0 4 3 | 5, 4 2 1 |
| – 1 9 | – 8 5 |
| 8, 5 9 1 | 6, 8 1 0 |
| –2, 7 8 6 | –2, 3 9 7 |

| □ □ □ □ | □ □ □ □ |
|---|---|
| (–3) (27) (27) (27) (30) | ( ) ( ) ( ) ( ) ( ) |
| 9, 0 0 0 | 7, 1 4 6 |
| –1, 2 3 4 | – 2 8 |
| –4, 5 6 7 | –1, 5 3 7 |
| – 9 8 | –2, 0 8 9 |

| □ □ □ □ | □ □ □ □ |
|---|---|
| ( ) ( ) ( ) ( ) ( ) | (–3) (27) (27) (27) (30) |
| 8, 4 2 9 | – 7 5 |
| – 7 3 1 | 8, 6 5 4 |
| –5, 0 4 6 | – 9 0 6 |
| –1, 1 5 7 | –2, 7 1 2 |

| □ □ □ □ | □ □ □ □ |
|---|---|
| ( ) ( ) ( ) ( ) ( ) | ( ) ( ) ( ) ( ) ( ) |
| – 5 6 | – 4 3 |
| 7, 2 7 3 | 9, 5 1 0 |
| – 5 2 7 | – 1 8 6 |
| –1, 3 0 8 | –4, 2 7 2 |

| □ □ □ □ | □ □ □ □ |
|---|---|
| (–3) (27) (27) (27) (30) | ( ) ( ) ( ) ( ) ( ) |
| – 2 9 | – 7 2 |
| – 5 6 4 | – 6 4 3 |
| –4, 0 7 5 | –1, 1 2 5 |
| 8, 7 9 3 | 9, 0 8 6 |

Combination problems of Patterns 1, 2, 3, 4, and 5

Since all 5 patterns are used, extra problems are given to help you see the various patterns.

5
8
– 6 (Interchange 5, Clear 1)
7
– 8 (Interchange 5, Clear 3)
– 2 (Clear 5, Press 3)

7
5
– 7 (Interchange 5, Clear 2)
9
– 9 (Interchange 5, Clear 4)
– 1 (Clear 5, Press 4)

9
8
– 6 (Available)
2
– 6 (Interchange 5, Clear 1)
– 3 (Clear 5, Press 2)

8
6
– 9 (Interch. 5, Clear 4)
– 1 (Clear 5, Press 4)
7
– 6 (Interch. 5, Clear 1)

9
8
– 6 (Avail.)
– 6 (Interch. 5, Clear 1)
9
– 7 (Interch. 5, Clear 2)

7
6
– 8 (Interch. 5, Clear 3)
– 2 (Clear 5, Press 3)
8
– 3 (Clear 10, Press 7)

7
9
– 3 (Clear 5, Press 2)
– 4 (Clear 10, Press 6)
5
– 9 (Interch. 5, Clear 4)

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 | 9 | 7 | 6 | 2 | 8 | 8 |
| 6 | 9 | 8 | 9 | 9 | 5 | 8 |
| 9 | 9 | 9 | 8 | 8 | 3 | 8 |
| − 6 | − 8 | − 9 | − 5 | − 7 | − 8 | − 2 |
| − 8 | − 7 | − 1 | − 7 | − 6 | − 5 | − 3 |
| − 3 | − 5 | − 7 | − 6 | − 2 | − 1 | − 4 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | 9 | 7 | 8 | 5 | 6 | 4 |
| − 3 | − 7 | − 4 | − 2 | − 1 | − 5 | − 3 |
| 9 | 8 | 9 | 8 | 7 | 4 | 9 |
| − 5 | − 1 | − 6 | − 3 | − 2 | − 1 | − 4 |
| 7 | 5 | 8 | 9 | 6 | 9 | 2 |
| − 9 | − 8 | − 7 | − 5 | − 3 | − 4 | − 4 |

## SUBTRACTION STAGE 4

As in addition, Subtraction Stage 4 is intended to accelerate the speed of calculation and to internalize the arithmetical lessons learned at the earlier stages. As in addition, students must be taught to combine subtrahends before proceeding with an operation. If the students have not reached negative numbers in their number theory, fourth stage subtraction provides an excellent opportunity to introduce this concept. Initially, students should be given examples of problems with multiple subtrahends and asked to subtract the subtrahends one at a time. This should be followed by asking the students to subtract while combining the subtrahends. Once they see that the result is the same, they will begin to understand the rationale of negative numbers.

| | (B) | (A) |
|---|---|---|
| (-2) | (18) | (20) |
| | 8 | 7 |
| - | 1 | 9 |
| - | 3 | 7 |

For example, in column (A) above, the students should first subtract 9 from 27 (Trade-in number 20 + addend 7), getting 18 as a partial result. They should then subtract 7 from 18 and reach the answer of 11.

This exercise would be followed by subtracting 16 from 27 (Trade-in 20 + addend 7) and reaching the ‚same final result. Finally, it should be noted that 27 (Trade-in 20 + addend 7) - 9 - 7 is really 27-(9+7) and that -9 -7 = -16. These facts should be reinforced with demonstrations on the HYPAI Chisanbop ™ and Decimal Number Stairs.

Before proceeding to Subtraction Stage 4, certain mental calculation drill exercises must be performed.

Just as Addition Stage 4 presumed that a student could mentally calculate any two single digit addends, Subtraction Stage 4 now presumes that the student can calculate any two single digits, whether they come from addends, subtrahends, or both.

The following mental calculation drill exercises are divided into three cases: In case I, both digits are subtrahends. In case II, the upper number is part of an addend and the lower number is part of a subtrahend. Case III is used when the upper number is from a subtrahend and the lower number is from an addend. The case where both numbers are from addends has already been practiced for Addition Stage 4.

# SUBTRACTION STAGE 4

## Mental Calculation Drill

### Case I

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| - 0 | - 0 | - 0 | - 0 | - 0 | - 0 | - 0 | - 0 | - 0 | - 0 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| - 1 | - 1 | - 1 | - 1 | - 1 | - 1 | - 1 | - 1 | - 1 | - 1 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| - 2 | - 2 | - 2 | - 2 | - 2 | - 2 | - 2 | - 2 | - 2 | - 2 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| - 3 | - 3 | - 3 | - 3 | - 3 | - 3 | - 3 | - 3 | - 3 | - 3 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| - 4 | - 4 | - 4 | - 4 | - 4 | - 4 | - 4 | - 4 | - 4 | - 4 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| - 5 | - 5 | - 5 | - 5 | - 5 | - 5 | - 5 | - 5 | - 5 | - 5 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| - 6 | - 6 | - 6 | - 6 | - 6 | - 6 | - 6 | - 6 | - 6 | - 6 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| - 7 | - 7 | - 7 | - 7 | - 7 | - 7 | - 7 | - 7 | - 7 | - 7 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| - 8 | - 8 | - 8 | - 8 | - 8 | - 8 | - 8 | - 8 | - 8 | - 8 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| - 9 | - 9 | - 9 | - 9 | - 9 | - 9 | - 9 | - 9 | - 9 | - 9 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |

## Mental Calculation Drill

### Case II

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |
| - 0 | - 1 | - 2 | - 3 | - 4 | - 5 | - 6 | - 7 | - 8 | - 9 |

## Mental Calculation Drill

### Case III

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| - 0 | - 0 | - 0 | - 0 | - 0 | - 0 | - 0 | - 0 | - 0 | - 0 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| - 1 | - 1 | - 1 | - 1 | - 1 | - 1 | - 1 | - 1 | - 1 | - 1 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| - 2 | - 2 | - 2 | - 2 | - 2 | - 2 | - 2 | - 2 | - 2 | - 2 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| - 3 | - 3 | - 3 | - 3 | - 3 | - 3 | - 3 | - 3 | - 3 | - 3 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| - 4 | - 4 | - 4 | - 4 | - 4 | - 4 | - 4 | - 4 | - 4 | - 4 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| - 5 | - 5 | - 5 | - 5 | - 5 | - 5 | - 5 | - 5 | - 5 | - 5 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| - 6 | - 6 | - 6 | - 6 | - 6 | - 6 | - 6 | - 6 | - 6 | - 6 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| - 7 | - 7 | - 7 | - 7 | - 7 | - 7 | - 7 | - 7 | - 7 | - 7 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| - 8 | - 8 | - 8 | - 8 | - 8 | - 8 | - 8 | - 8 | - 8 | - 8 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| - 9 | - 9 | - 9 | - 9 | - 9 | - 9 | - 9 | - 9 | - 9 | - 9 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

## SUBTRACTION STAGE 4

Now that you have completed the mental calculation exercises on the previous pages, you can now proceed to Subtraction Stage 4. The rules are very simple: 1) Make the HYPAI Power Trades. 2) Calculate column-by-column. 3) When you calculate column-by-column, mentally calculate the value of two digits and then press or clear the results of this calculation.

In the following examples, all the final results will be positive. We will deal with negative answers in an optional section later.

| □ | □ | □ | □ | |
|---|---|---|---|---|
| (-3) | (27) | (27) | (27) | (30) |
| - | 3, | 0 | 5 | 7 |
| - | 2, | 6 | 3 | 4 |
| | 5, | 6 | 9 | 8 |
| | 4, | 2 | 3 | 2 |
| - | 1, | 9 | 3 | 7 |
| | 6, | 8 | 5 | 2 |
| | | | | 4 |

Looking at the problem above, the 3 subtrahends require the Power Trade numbers to be 30, 27 and -3. Begin the calculation by registering 30 and then clearing -11 for the upper two digits in the units' column.

We then add 10 for the middle two digits, and then clear 5 for the last two digits. We would write 4 below the line for the answer. We would then carry over the 2 registered on the left hand to the tens' column or transfer directly the 2 to the right hand. After transferring the 2 from the units' column, another 27 would have to be pressed from the Power Trade.

Once having registered the carry-over and the Power Trades on our fingers, we would proceed to calculate the rest of the tens' column as we did the units' column, pairing and calculating two digits before we registered them on our fingers. The same procedure would be followed in the hundreds' and thousands' column. At this point students can begin to take another shortcut.

If they are secure in understanding and remembering the numbers which are traded, they do not have to write them down. They only have to remember to register the proper number on their fingers as they begin to calculate each column.

Moving from third to fourth stage subtraction is a gradual process. If students have difficulty subtracting certain number combinations, they can always use the Stage 3 manipulations until they attain some comfort in Stage 4 manipulations.

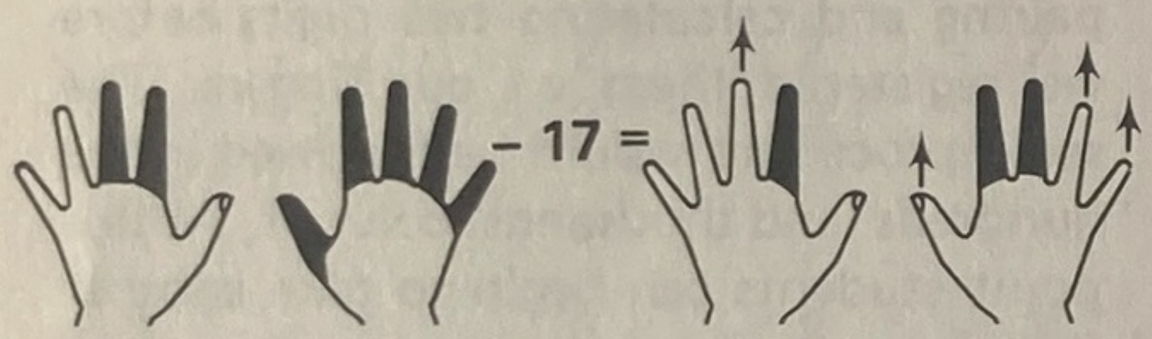
In the Subtraction Stage 4, the way to clear a number always depends on what fingers are already pressed.

The following example illustrates three different ways to subtract 17.

**Illustration Example:**
**(Subtracting the number 17)**

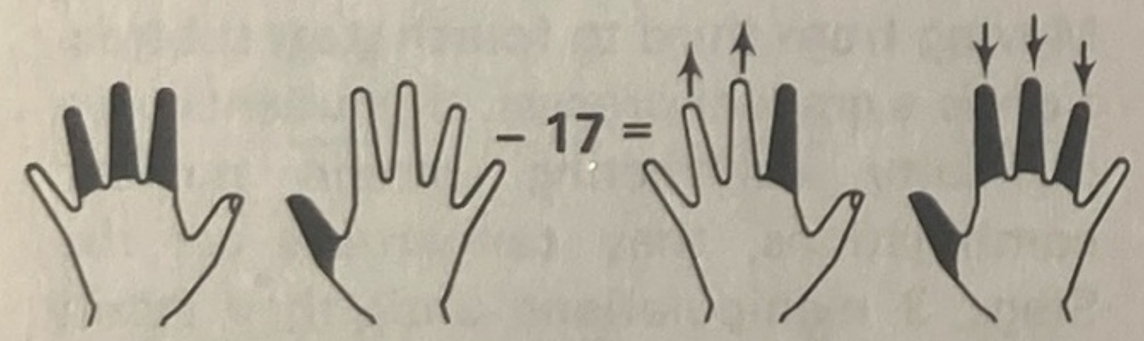
*Case 1: (–17):* The required fingers are available for clearing. When 17 is subtracted from the numbers 17, 18, 19, 27, 28, 29, 37, 38, 47, 48, or 49 etc. the fingers required to subtract 17 are available for clearing.

*29 is already pressed* (Clear 10 and Clear 7 simultaneously)




*Case 2: (–20+3):* When 17 is subtracted from the numbers 25, 26, 35, 36, 45, or 46, etc. we must use the (–20+3)–clear 20, press 3–manipulation.

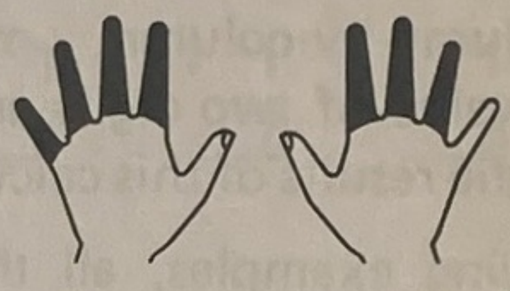
*35 is already pressed* (Clear 20, Press 3 simultaneously)



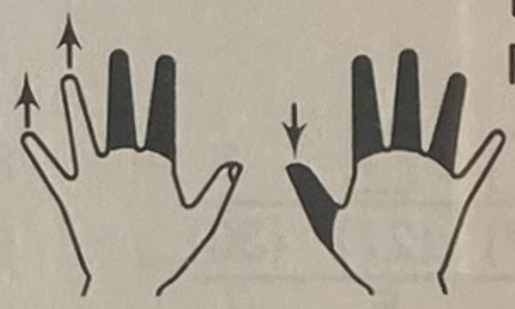

*Case 3: (Interchange 15–2)*

When the right hand registers 2, 3, or 4, we must use the Interchange 15–2 manipulation. Interchange 15 means clear 20, press 5 (–20+5) simultaneously.

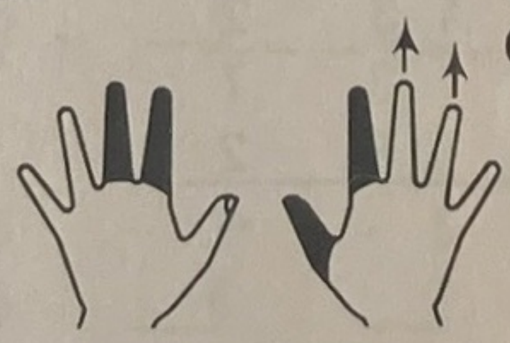
*43 is already pressed*




A. (Interchange 15 through Clear 20, Press 5 manipulation)



B. (and simultaneously Clear 2)



## SUBTRACTION STAGE 4 PATTERN CHART

Subtraction Stage 4 manipulations are combinations of following mathematical patterns employed in *subtracting the sum of numbers -11 to 18-*. We can still employ the 3rd stage subtraction manipulations for the sum of two numbers less than 10.

The fingers already pressed on the right hand determine which pattern to use.

| Pattern / Subtrahends | Pattern 1 | Pattern 2 | Pattern 3 | Pattern 4 | Pattern 5 |
|---|---|---|---|---|---|
| -11 | -11 Available for Clearing | | -15+4 | -20+9 (Interch. 15+4)* | |
| -12 | -12 Available for Clearing | | -15+3 | -20+8 (Interch. 15+3)* | |
| -13 | -13 Available for Clearing | | -15+2 | -20+7 (Interch. 15+2)* | |
| -14 | -14 Available for Clearing | | -15+1 | -20+6 (Interch. 15+1)* | |
| -15 | -15 Available for Clearing | Interchange -15 (-20+5) | | | |
| -16 | -16 Available for Clearing | | | -20+4 | Interch. 15-1 |
| -17 | -17 Available for Clearing | | | -20+3 | Interch. 15-2 |
| -18 | -18 Available for Clearing | | | -20+2 | Interch. 15-3 |

*The instructions in the parentheses can be used as a temporary manipulation until the student gets used to the more sophisticated maneuver as instructed above the parentheses.

## EXERCISES

Pattern 1 (Clear Directly)

| | | | |
|---|---|---|---|
| 7<br>5<br>4<br>7 | –11 Available for Clearing | 8<br>6<br>2<br>2 | –4 Available for Clearing |
| 7<br>6<br>8<br>4 | –12 Available for Clearing | 9<br>7<br>8<br>7 | –15 Available for Clearing |
| 8<br>9<br>7<br>5 | –12 Available for Clearing | 9<br>9<br>8<br>9 | –17 Available for Clearing |
| 9<br>8<br>8<br>8 | –16 Available for Clearing | 6<br>8<br>7<br>7 | –14 Available for Clearing |

Pattern 2 (Interchange 5 (Clear 20, Press 5))

| | | | |
|---|---|---|---|
| 1<br>9<br>3<br>7<br>6<br>9 | Interchange 15 (–20+5) | 6<br>5<br>5<br>5<br>8<br>7 | Interchange 15 (–20+5) |
| 6<br>1<br>9<br>8<br>9<br>6 | Interchange 15 (–20+5) | 8<br>7<br>9<br>9<br>7<br>8 | Interchange 15 (–20+5) |
| 7<br>6<br>4<br>7<br>6<br>9 | Interchange 15 (–20+5) | 8<br>2<br>9<br>3<br>8<br>7 | Interchange 15 (–20+5) |

| | | | |
|---|---|---|---|
| 8<br>5<br>4<br>6<br>9<br>6 | Interchange 15 (–20+5) | 2<br>7<br>9<br>5<br>7<br>8 | Interchange 15 (–20+5) |

Pattern 3 (Clear 15, Press ___)

| | |
|---|---|
| 6<br>6<br>7<br>6<br>7<br>7 | Clear 15, Press 1 |
| 1<br>9<br>8<br>7<br>8<br>5 | Clear 15, Press 2 |
| 8<br>6<br>9<br>3<br>7<br>5 | Clear 15, Press 3 |
| 7<br>9<br>5<br>4<br>2<br>9 | Clear 15, Press 4 |
| 8<br>9<br>3<br>7<br>5<br>9 | Clear 15, Press 1 |

7
7
9

Clear 15,
Press 2

9
1
7
6

Clear 15,
Press 3

5
4
9
4
9

(Interchange 15,
Clear 2)

Pattern 4 (Clear 20, Press ___)

8
9
6
3
9

(Clear 20,
Press 2)

7
6
9
3
8

(Clear 20,
Press 3)

8
6
5
1
7

(Clear 20,
Press 4)

5
6
4
8
9

(Clear 20,
Press 6)

7
5
6
4
6

(Clear 20,
Press 7)

8
7
2
5
9

(Clear 20,
Press 7)

3
5
6
7
6

(Clear 20,
Press 8)

5
5
1
9
4

(Clear 20,
Press 9)

8
6
2
8
8

(Interchange 15,
Clear 2)

7
7
5
4
9

(Interchange 15,
Clear 3)

6
2
8
7
7

(Interchange 15,
Clear 1)

6
8
6
4
9

(Interchange 15,
Clear 2)

Pattern 5 (Interchange 15, Clear ___)

4
5
5
7
8

(Interchange 15,
Clear 1)

3
9
5
7
9

(Interchange 15,
Clear 3)

Pattern 1 (Clear directly)

| | | | | | | |
|---|---|---|---|---|---|---|
| 7 | 8 | 7 | 9 | 8 | 9 | 9 |
| 5 | 6 | 6 | 7 | 9 | 9 | 8 |
| 4 | 6 | 8 | 8 | 7 | 6 | 8 |
| 7 | 6 | 4 | 3 | 5 | 9 | 8 |

Pattern 2 (Interchange 15 (Clear 20, Press 5))

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 6 | 6 | 8 | 7 | 8 | 8 |
| 9 | 5 | 1 | 7 | 6 | 2 | 5 |
| 3 | 5 | 9 | 9 | 4 | 9 | 4 |
| 7 | 5 | 8 | 9 | 7 | 3 | 6 |
| 6 | 8 | 9 | 7 | 6 | 8 | 9 |
| 9 | 7 | 6 | 8 | 9 | 7 | 6 |

Pattern 3 (Clear 15, Press ___)

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | 8 | 1 | 7 | 8 | 9 | 7 |
| 6 | 9 | 9 | 7 | 6 | 9 | 9 |
| 7 | 3 | 8 | 9 | 8 | 1 | 5 |
| 6 | 7 | 7 | 4 | 3 | 7 | 4 |
| 7 | 5 | 8 | 7 | 7 | 6 | 2 |
| 7 | 9 | 5 | 6 | 5 | 6 | 9 |

Pattern 4 (Clear 20, Press ___)

| | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 7 | 8 | 5 | 7 | 8 | 5 |
| 9 | 6 | 6 | 6 | 5 | 7 | 5 |
| 6 | 9 | 5 | 4 | 6 | 2 | 1 |
| 3 | 3 | 1 | 8 | 4 | 5 | 9 |
| 9 | 8 | 7 | 5 | 6 | 9 | 4 |
| 9 | 9 | 9 | 9 | 7 | 4 | 7 |

Pattern 5 (Interchange 15, Clear ___)

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | 5 | 8 | 6 | 8 | 5 | 7 |
| 5 | 4 | 6 | 2 | 5 | 8 | 7 |
| 5 | 9 | 6 | 6 | 1 | 6 | 5 |
| 7 | 4 | 4 | 7 | 8 | 4 | 4 |
| 8 | 9 | 9 | 7 | 8 | 9 | 9 |
| 8 | 8 | 9 | 9 | 9 | 8 | 9 |

## PRACTICE EXERCISES

### HYPAI Power Trading

### Calculate Column-by-Column After HYPAI Power Trades

| □ | □ | □ | □ | |
|---|---|---|---|---|
| (–2) | (18) | (18) | (18) | (20) |
| | 6, | 9 | 3 | 0 |
| | 8, | 0 | 2 | 6 |
| | –4, | 3 | 9 | 5 |
| | –1, | 5 | 8 | 4 |

| □□ | □ | □ | |
|---|---|---|---|
| | | | |
| 7, | 2 | 4 | 9 |
| 7, | 6 | 9 | 8 |
| –1, | 9 | 3 | 5 |
| –2, | 7 | 0 | 6 |

| □□ | □ | □ | |
|---|---|---|---|
| | | | |
| 9, | 0 | 6 | 7 |
| 9, | 8 | 5 | 4 |
| –2, | 9 | 5 | 7 |
| –1, | 9 | 8 | 3 |

| □ | □ | □ | □ | |
|---|---|---|---|---|
| (–2) | (18) | (18) | (18) | (20) |
| | 8, | 6 | 4 | 6 |
| | 7, | 5 | 0 | 2 |
| | –6, | 9 | 3 | 7 |
| | –2, | 7 | 5 | 1 |

| □□ | □ | □ | |
|---|---|---|---|
| | | | |
| 7, | 6 | 9 | 0 |
| 8, | 7 | 5 | 4 |
| –2, | 9 | 3 | 2 |
| –9, | 2 | 3 | 7 |

| □□ | □ | □ | |
|---|---|---|---|
| | | | |
| 8, | 7 | 1 | 4 |
| 3, | 0 | 2 | 5 |
| –6, | 5 | 1 | 3 |
| –2, | 9 | 8 | 1 |

| □ | □ | □ | □ | |
|---|---|---|---|---|
| (–2) | (18) | (18) | (18) | (20) |
| | –2, | 6 | 9 | 4 |
| | –6, | 9 | 5 | 2 |
| | 7, | 8 | 6 | 9 |
| | 9, | 0 | 8 | 9 |

□□□□

| | | | |
|---|---|---|---|
| | −4, | 3 | 2 | 7 |
| | −6, | 9 | 3 | 6 |
| | 9, | 0 | 1 | 5 |
| | 7, | 5 | 6 | 3 |

□□□□

| | | | |
|---|---|---|---|
| | −1, | 6 | 7 | 5 |
| | −3, | 4 | 0 | 5 |
| | 6, | 9 | 3 | 1 |
| | 7, | 5 | 2 | 3 |

□□□□

| | | | |
|---|---|---|---|
| | −1, | 5 | 8 | 6 |
| | 3, | 6 | 0 | 7 |
| | −4, | 9 | 3 | 2 |
| | 8, | 0 | 4 | 1 |

□□□□

| | | | |
|---|---|---|---|
| | −2, | 1 | 4 | 6 |
| | 3, | 0 | 6 | 7 |
| | −4, | 9 | 5 | 3 |
| | 8, | 6 | 2 | 1 |

□□□□

| (−2) | (18) | (18) | (18) | (20) |
|---|---|---|---|---|
| | 6, | 0 | 4 | 3 |
| | −1, | 8 | 6 | 9 |
| | 8, | 7 | 5 | 2 |
| | −3, | 1 | 6 | 2 |

□□□□

| | | | |
|---|---|---|---|
| | 9, | 5 | 7 | 2 |
| | −6, | 2 | 9 | 4 |
| | 7, | 0 | 3 | 1 |
| | −1, | 8 | 4 | 0 |

□□□□

| (−2) | (18) | (18) | (18) | (20) |
|---|---|---|---|---|
| | −5, | 2 | 1 | 6 |
| | 9, | 8 | 4 | 8 |
| | −6, | 7 | 9 | 7 |
| | 9, | 0 | 3 | 8 |

□□□□

| (−2) | (18) | (18) | (18) | (20) |
|---|---|---|---|---|
| | 5, | 2 | 1 | 0 |
| | −1, | 8 | 9 | 3 |
| | 6, | 6 | 7 | 8 |
| | −2, | 9 | 3 | 5 |

□□□□

| | | | |
|---|---|---|---|
| | 6, | 0 | 4 | 3 |
| | −7, | 2 | 2 | 8 |
| | 9, | 8 | 9 | 1 |
| | −4, | 0 | 8 | 7 |

□□□□

| | | | |
|---|---|---|---|
| | −2, | 7 | 5 | 4 |
| | 7, | 6 | 3 | 9 |
| | −8, | 0 | 2 | 6 |
| | 8, | 9 | 3 | 1 |

□□□□

| | | | |
|---|---|---|---|
| | −1, | 1 | 2 | 2 |
| | 7, | 6 | 5 | 8 |
| | −3, | 4 | 0 | 6 |
| | 6, | 2 | 3 | 5 |

□□□□

| (−2) | (18) | (18) | (18) | (20) |
|---|---|---|---|---|
| | 7, | 7 | 5 | 1 |
| | −4, | 6 | 9 | 3 |
| | 8, | 5 | 0 | 6 |
| | −7, | 0 | 9 | 8 |

□□□□

| | | | |
|---|---|---|---|
| | 6, | 9 | 1 | 4 |
| | −2, | 7 | 8 | 5 |
| | 9, | 0 | 0 | 6 |
| | −7, | 9 | 3 | 4 |

□□□□

| | | | |
|---|---|---|---|
| | 2, | 6 | 3 | 7 |
| | −4, | 8 | 1 | 5 |
| | 9, | 0 | 3 | 6 |
| | −1, | 5 | 9 | 3 |

□□□□

| | | | |
|---|---|---|---|
| | 5, | 0 | 7 | 3 |
| | −8, | 5 | 2 | 6 |
| | 9, | 2 | 0 | 1 |
| | −1, | 5 | 5 | 8 |

□□□□

| (−2) | (18) | (18) | (18) | (20) |
|---|---|---|---|---|
| | −6, | 7 | 5 | 4 |
| | 7, | 5 | 9 | 3 |
| | − | 2 | 9 | 8 |
| | 8, | 0 | 2 | 7 |

□□□□

| | | | |
|---|---|---|---|
| | 1, | 5 | 3 | 6 |
| | −4, | 0 | 2 | 4 |
| | 9, | 6 | 7 | 5 |
| | −1, | 5 | 3 | 7 |

□□□□

| (−2) | (18) | (18) | (18) | (20) |
|---|---|---|---|---|
| | −6, | 2 | 5 | 9 |
| | 9, | 3 | 0 | 6 |
| | −3, | 4 | 2 | 7 |
| | 8, | 0 | 3 | 7 |

□□□□

| | | | |
|---|---|---|---|
| | − | 6 | 0 | 6 |
| | 1, | 0 | 7 | 5 |
| | −2, | 9 | 3 | 4 |
| | 9, | 2 | 4 | 2 |

□□□□

| | | | |
|---|---|---|---|
| | −1, | 2 | 4 | 6 |
| | | 6 | 8 | 3 |
| | −2, | 9 | 5 | 7 |
| | 6, | 0 | 8 | 2 |

Transfer carry-over number directly from left hand to right hand.

| □ | □ | □ | □ | |
|---|---|---|---|---|
| (–3) | (27) | (27) | (27) | (30) |
| | 6, | 7 | 5 | 9 |
| | 8, | 2 | 6 | 1 |
| | 5, | 0 | 5 | 2 |
| | –4, | 9 | 3 | 9 |
| | –5, | 4 | 0 | 8 |
| | –6, | 8 | 2 | 1 |

| □ | □ | □ | □ | |
|---|---|---|---|---|
| (–3) | (27) | (27) | (27) | (30) |
| | –2, | 8 | 3 | 1 |
| | 5, | 9 | 4 | 3 |
| | 9, | 0 | 7 | 5 |
| | – | 7 | 6 | 3 |
| | – | | 8 | 4 |
| | 8, | 1 | 6 | 5 |

| □ | □ | □ | □ |
|---|---|---|---|
| | 5, | 0 | 6 | 3 |
| | –7, | 8 | 2 | 6 |
| | –1, | 3 | 1 | 0 |
| | 4, | 5 | 2 | 4 |
| | 8, | 7 | 5 | 7 |
| | –3, | 9 | 5 | 8 |

| □ | □ | □ | □ |
|---|---|---|---|
| | 6, | 7 | 5 | 3 |
| | –1, | 0 | 8 | 6 |
| | –4, | 5 | 2 | 2 |
| | 9, | 8 | 7 | 0 |
| | 3, | 6 | 1 | 4 |
| | –3, | 9 | 6 | 5 |

| □ | □ | □ | □ |
|---|---|---|---|
| | 9, | 5 | 0 | 6 |
| | –4, | 7 | 8 | 8 |
| | – | 6 | 4 | 3 |
| | –2, | 2 | 7 | 0 |
| | 5, | 1 | 8 | 9 |
| | 6, | 3 | 5 | 2 |

| □ | □ | □ | □ |
|---|---|---|---|
| | 5, | 2 | 0 | 8 |
| | – | | | 6 |
| | 7, | 0 | 3 | 9 |
| | –4, | 5 | 2 | 8 |
| | 9, | 6 | 5 | 4 |
| | – | 3 | 6 | 9 |

## INTERNALIZATION

As the student practices these Subtraction Stage 4 problems and finds himself repeatedly correct in his calculations, he should be reaching a stage where he can begin to abandon the use of his fingers. He has indeed already internalized the arithmetical operations, but now must begin to eliminate the sense of touch so that he can reach a stage of even more rapid calculation. This elimination of the tactile sense should not begin until the student is completely secure in the knowledge of the arithmetical process. Once he knows he can always succeed, he can make the next move in two steps.

The first step is to raise the hands off the desk or table and let the palms of the hands face each other. Initially, when a pressing manipulation is called for, the fingers can be pressed all the way to the palms. The tactile sense has been weakened since it does not have a hard object to press against the fingers.

The second step in abandoning the fingers is to only slightly curl the fingers when a pressing manipulation is called for. Eventually the fingers will be abandoned entirely.

## COMPLEMENTARY NUMBERS

Complementary numbers present a method for finding the answer to a subtraction problem by using addition. It is easy and incredibly rapid especially when there are multiple addends interspersed among several subtrahends. This method is not crucial

to Chisanbop (TM) theory and should be treated as an optional lesson. However, before proceeding to the method several words of warning should be given to educators.

Many teachers have found that the complementary number method of subtraction is a remarkable system for teaching students needing remedial help. There have been countless reports of students who could never learn subtraction repeatedly getting the correct results. However, while the method shows a youngster how to get the right answer, he will never understand the nature of subtraction using this method. Thus with slower and handicapped students, it is suggested that this method be taught *only* after *all* other attempts to teach subtraction have been tried and failed. After all, the purpose of teaching these students is not only to show them how to get an answer, but also to lead them to understand the meaning and nature of the arithmetical operations.

The complementary method is presented in three parts. First it shows how to find an answer when there is one subtrahend. Then it shows how to find an answer for several subtrahends in mixed operation problems. Finally a rationale as to why this system works is shown.

## How to Use Complementary Numbers

Before showing how to use complementary numbers, we must first define them.

The complement of 0 is 9 and the complement of 9 is 0.

The complement of 1 is 8 and the complement of 8 is 1.

The complement of 2 is 7 and the complement of 7 is 2.

The complement of 3 is 6 and the complement of 6 is 3.

The complement of 4 is 5 and the complement of 5 is 4.

Thus if any two single-digit numbers add up to 9, they are complements of one another.

There are four simple rules to subtracting with complementary numbers.

### Single Subtrahend

1) Rewrite the subtrahend in terms of its complements. Once having seen how this method works, you should not physically rewrite the subtrahends in terms of its complements, but make the transformation mentally.
2) Add a +1 to the units' column.
3) Place a -1 one column to the left of the last digit of the subtrahend.
4) Add the minuend and the complementary numbers of the subtrahend. Don't forget the +1 and -1 from steps 2 and 3. Don't be upset by carry-over numbers—they often occur with this method. With the exception of the -1 in the far left-hand column, this entire subtraction problem was performed using addition.

```
   7 3 2 6
-  4 7 8 1  <-- Subtrahend
```

becomes

```
-1          +1
   7 3 2 6
+  5 2 1 8  <-- Complement
   2 5 4 5      of Subtrahend
```

It should be pointed out that this type of problem is not presented in traditional texts only because there is no easy way to handle it traditionally. However, this kind of problem does exist in real life. One example would be the deli owner who pays tradesmen and receives cash payments, all in and out of the same register.

## Why Complementary Numbers Work.

The following is a twofold demonstration of why we can use complementary numbers. In each case we shall use our original problem, i.e. 7,326–4,781. The first demonstration is visual; the second is based upon HYPAI Power Trading.

A. If we add and then subtract 10,000 to our problem, we all know that none of the values will change; i.e. 7,326–4,781 = 7,326–4,781+10,000–10,000. This essentially is what happens using the complementary number method. The subtraction of 10,000 is readily apparent in the –1 we placed in the far left-hand column. The adding of 10,000 must be examined more closely.

10,000=9,000+900+90+10. If we try to draw a picture of this number, letting the boxes in the far right column represent units and those second from the right represent tens, etc. it will look like diagram 1. If we take the picture from diagram 1, and shade in the value of the subtrahend (–4,781) with light shading ( ▨ ) it will look like diagram 2. Finally, if we put dark shading ( ■ ) in every other box except the one at the top of the units' column, it will look like diagram 3. If you count the darkly shaded boxes, you will find that they represent the

**DIAGRAM 1**

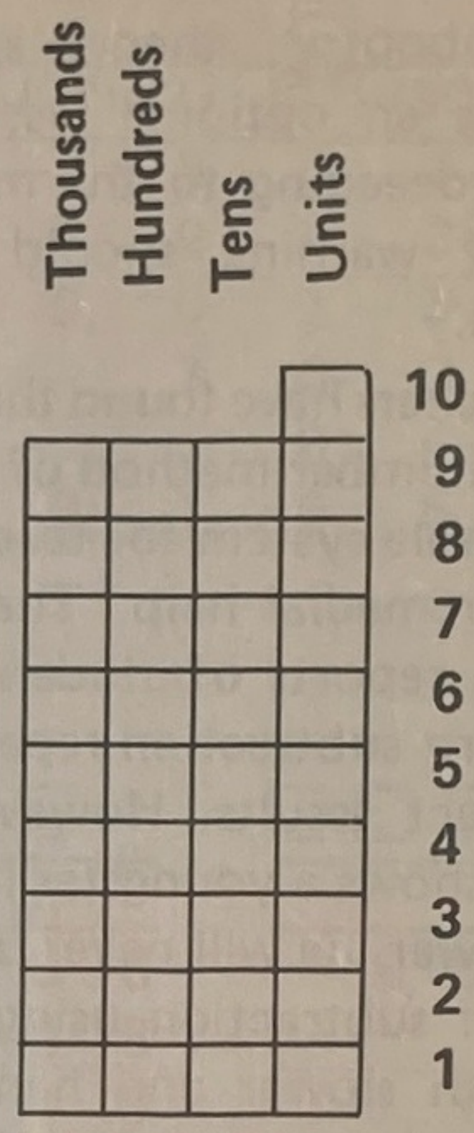



**DIAGRAM 2**

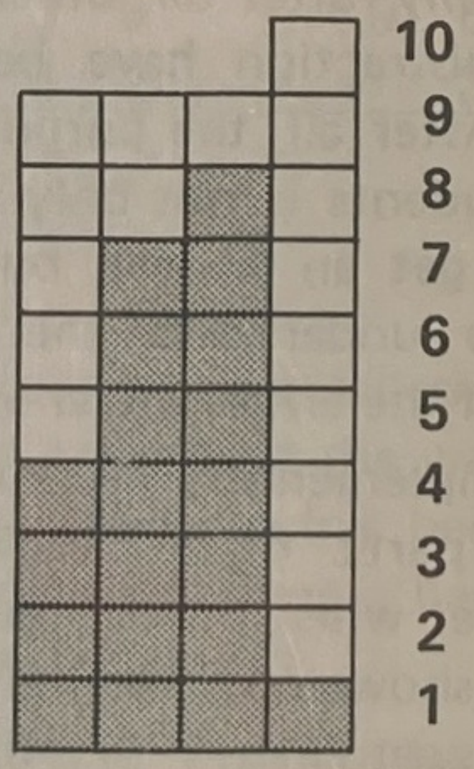
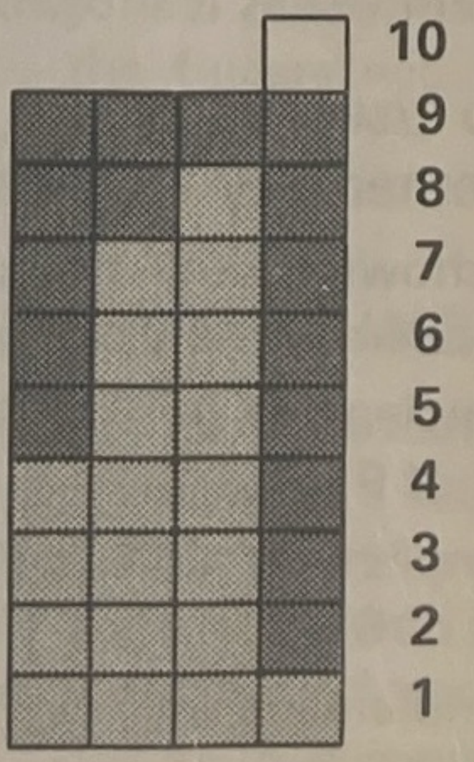



**DIAGRAM 3**



the complements (+5,218) of the subtrahend, (-4,781). We really do add 10,000, but do it in the combined operation of also subtracting the subtrahend. To combine adding 10,000 and subtracting the subtrahend, we merely add the complements. The blank box in the units' column was the +1 we always add to the units before starting.

B. Proof Using HYPAI Trading. Once more restate the original problem:

| | | | |
|---|---|---|---|
| 7, | 3 | 2 | 6 |
| -4, | 7 | 8 | 1 |

If we make HYPAI Power Trades, our problem will look like this.

| | | | | |
|---|---|---|---|---|
| (-1) | (9) | (9) | (9) | (10) |
| | 7, | 3 | 2 | 6 |
| | -4, | 7 | 8 | 1 |

We make two more final adjustments changing the +10 in the units' column into +9+1 and removing the parentheses of the Power Trade. Finally, the problem looks like this.

| | | | | |
|---|---|---|---|---|
| | | | | 1 |
| -1 | 9 | 9 | 9 | 9 |
| | 7, | 3 | 2 | 6 |
| | -4, | 7 | 8 | 1 |

If we now examine the problem, we find that in each column we are asked to add a 9 to the minuend and then subtract the digit of the subtrahend. This procedure can be shortened by merely adding the complement. Thus, in the units' column, instead of adding the 9 from the Power Trade and subtracting the 6 from the subtrahend, we abbreviate the procedure by adding 3 (the complement of 6). The +1 at the top of the units' column and -1 in the ten thousands' column were the numbers added as initially stated in steps 2 and 3 in the rules for Complementary Numbers.

### Multiple Subtrahends

This method is especially rapid when there is more than one subtrahend or when the problem has mixed operations. The rule for multiple subtrahends is substantially the same. Each subtrahend must be rewritten in terms of its complement. (Note the value of the 3rd subtrahend is *0*27 and its complement is thus *9*72.) The only change is that a +1 must be placed in the units' column for each subtrahend and a -1 must be placed in the new column on the left for each subtrahend. Hence for the problem shown we have added +3 to the units and -3 to the thousands.

| | | | |
|---|---|---|---|
| +5 | 3 | 7 | |
| +2 | 2 | 4 | |
| -4 | 6 | 3 | Subtrahends |
| -5 | 2 | 8 | |
| +8 | 6 | 6 | |
| -0 | 2 | 7 | |

becomes

| | | | |
|---|---|---|---|
| -3 | | +3 | |
| +5 | 3 | 7 | |
| +2 | 2 | 4 | |
| +5 | 3 | 6 | Complements |
| +4 | 7 | 1 | of Subtrahends |
| +8 | 6 | 6 | |
| +9 | 7 | 2 | |
| 6 | 0 | 9 | |

## PRACTICE EXERCISES
## TYPE 1

### COMPLEMENTARY NUMBERS

| | |
|---|---|
| 0 ⟷ 9 | 5 ⟷ 4 |
| 1 ⟷ 8 | 6 ⟷ 3 |
| 2 ⟷ 7 | 7 ⟷ 2 |
| 3 ⟷ 6 | 8 ⟷ 1 |
| 4 ⟷ 5 | 9 ⟷ 0 |

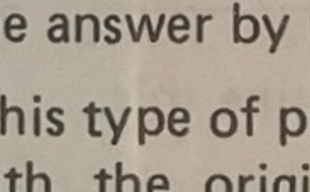
Add the minuend and complementary numbers of the subtrahends by column.

```
-1                +1        -1                +1
  60,403            72,416
- 27,819          - 16,937
-------           -------
       84                 
-1                +1        -1                +1
  86,402            61,073
- 58,947          - 26,985
-------           -------
       55                 
-1                +1        -1                +1
  80,196            68,236
- 47,325          - 42,827
-------           -------
       71                 
-1                +1        -1                +1
  62,389            51,708
- 27,415          - 38,296
-------           -------
       74                 
-1                +1        -1                +1
  90,010            48,704
- 68,437          - 27,893
-------           -------
-1                +1        -1                +1
  52,361            39,568
- 27,894          - 18,974
-------           -------
```

### COMPLEMENTARY NUMBERS
## TYPE 2

The method you have seen so far is consistent and should be used for all types of problems with slow learners. Consider the following problem:

```
  74,865
-     97
-------
```



```
-1                +1
  74,865
+99,902
-------
  74,768
```

Recognizing that the subtrahend 97 could be rewritten as 00,097 without changing its value, the complement of the subtrahend becomes 99,902. We transform the subtrahend using complements mentally, and find the answer by using addition.

Although this type of problem can be performed with the original format, some useless calculation was involved and therefore time was wasted. If you have students who require *one* consistent rule, use the first format. If you have students who are bright and able to discern various patterns, we recommend switching to the new format as seen below.

```
                  +1
  74,865
    -1
  - 97
-------
```

The -1, which we use to place in a new column to the left of the problem, we now place over the subtraction sign. You can now calculate this problem in the units' and tens' columns using complements; when you get to the hundreds' column, you perform the simple subtraction of 748–1. You thereby save yourself considerable computation.

## TYPE 3

With the new format, you occasionally run into problems such as the following.

(1)

```
          +1
  50,062
     -1
    - 87
  -------
      75
```

(2)

```
          +1
  50,062
     -1
    - 87
  -------
  49,975
```

(3)

```
          +1
  50,062
-1
-  ○○○87
  -------
  49,975
```

When attempting to calculate the hundreds' column, you are unable because there is no carry-over and the digit in the minuend is zero. If the students are capable of subtracting 1 from 500 directly, they can make the computation as seen in illustration (2). If the students do not understand this method, they must consider the subtrahend to be 00,087 and mentally calculate its complement to be 99,913 then proceed to finish the problem using complements in every column as illustrated in (3).

## TYPE 4

The illustration below shows a Type 4 problem, viz. 2 or more subtrahends.

```
            +2
  7532
    -1
   - 43
-1
- 756
  ----
```

Once again, the –1 should be placed directly over the subtraction sign in order to keep it in the proper place. In the illustrated problem, you should use the complements to find the answer in the units' and tens' columns. In the hundreds' column, you should subtract the –1 directly from the minuend and then add the complement from the second subtrahend.

In the thousands' column, you should subtract the –1 directly from the minuend.

Occasionally, in Type 4 problems, you must accomodate to the fact that you are not able to subtract a –1. The illustration below shows you how to make this accommodation.

```
    +2                          +2
  6003                        6003
    -1                          -1
   - 59                        - ○59
-1                          -1
- 758                       - 758
  ----                        ----
```

# PRACTICE EXERCISES

## COMPLEMENTARY NUMBERS

| | |
|---|---|
| 0 ⟷ 9 | 5 ⟷ 4 |
| 1 ⟷ 8 | 6 ⟷ 3 |
| 2 ⟷ 7 | 7 ⟷ 2 |
| 3 ⟷ 6 | 8 ⟷ 1 |
| 4 ⟷ 5 | 9 ⟷ 0 |

Add the minuend and complementary numbers of the subtrahends by column.

## TYPE 2

$$\begin{array}{r} {\scriptstyle +1} \\ 40,913 \\ {\scriptstyle -1} \\ -\ 7,346 \\ \hline 67 \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 55,607 \\ {\scriptstyle -1} \\ -\ 8,428 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 69,408 \\ {\scriptstyle -1} \\ -\ 787 \\ \hline 21 \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 89,734 \\ {\scriptstyle -1} \\ -\ 567 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 86,100 \\ {\scriptstyle -1} \\ -\ 9,824 \\ \hline 76 \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 12,169 \\ {\scriptstyle -1} \\ -\ 8,759 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 81,210 \\ {\scriptstyle -1} \\ -\ 186 \\ \hline 24 \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 67,814 \\ {\scriptstyle -1} \\ -\ 780 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 60,573 \\ {\scriptstyle -1} \\ -\ 9,099 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 98,427 \\ {\scriptstyle -1} \\ -\ 1,698 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 33,769 \\ {\scriptstyle -1} \\ -\ 423 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 52,063 \\ {\scriptstyle -1} \\ -\ 928 \\ \hline \end{array}$$

## TYPE 3

$$\begin{array}{r} {\scriptstyle +1} \\ 10,001 \\ {\scriptstyle -1} \\ -\ 724 \\ \hline 77 \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 60,275 \\ {\scriptstyle -1} \\ -\ 898 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 90,293 \\ {\scriptstyle -1} \\ -\ 706 \\ \hline 87 \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 40,317 \\ {\scriptstyle -1} \\ -\ 852 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 70,010 \\ {\scriptstyle -1} \\ -\ 89 \\ \hline 21 \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 20,036 \\ {\scriptstyle -1} \\ -\ 72 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 87,013 \\ {\scriptstyle -1} \\ -\ 79 \\ \hline 00 \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 25,034 \\ {\scriptstyle -1} \\ -\ 60 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 50,231 \\ {\scriptstyle -1} \\ -\ 516 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 10,045 \\ {\scriptstyle -1} \\ -\ 96 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 30,748 \\ {\scriptstyle -1} \\ -\ 960 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +1} \\ 61,037 \\ {\scriptstyle -1} \\ -\ 53 \\ \hline \end{array}$$

## TYPE 4

$$\begin{array}{r} {\scriptstyle +2} \\ 7,286 \\ {\scriptstyle -1} \\ -\ 1,699 \\ {\scriptstyle -1} \\ -\ 3,858 \\ \hline 29 \end{array}$$

$$\begin{array}{r} {\scriptstyle +2} \\ 6,040 \\ {\scriptstyle -1} \\ 2,785 \\ {\scriptstyle -1} \\ -\ 1,639 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +2} \\ 9,021 \\ {\scriptstyle -1} \\ -\ 987 \\ {\scriptstyle -1} \\ -\ 876 \\ \hline 58 \end{array}$$

$$\begin{array}{r} {\scriptstyle +2} \\ 7,316 \\ {\scriptstyle -1} \\ -\ 893 \\ {\scriptstyle -1} \\ -\ 760 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +2} \\ 7,034 \\ {\scriptstyle -1} \\ -\ 895 \\ {\scriptstyle -1} \\ -\ 73 \\ \hline 66 \end{array}$$

$$\begin{array}{r} {\scriptstyle +2} \\ 6,384 \\ {\scriptstyle -1} \\ -\ 59 \\ {\scriptstyle -1} \\ -\ 728 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +2} \\ 7,361 \\ {\scriptstyle -1} \\ -\ 1,809 \\ {\scriptstyle -1} \\ -\ 3,678 \\ \hline \end{array}$$

$$\begin{array}{r} {\scriptstyle +2} \\ 8,610 \\ {\scriptstyle -1} \\ -\ 789 \\ {\scriptstyle -1} \\ -\ 964 \\ \hline \end{array}$$

## Complementary Numbers Versus Missing Addends

The value of complementary numbers is that it offers a dynamic method of performing subtraction problems that require borrowing. Since the subtraction problem is now performed by means of addition, very rapid calculation is now possible.

Some books present another method for performing such a problem. Instead of finding differences they translate the problem into missing addends. The illustration below shows the way of translating the problem.

$$\begin{array}{r} 436 \\ -\,187 \\ \hline \end{array} \quad \text{becomes} \quad \begin{array}{r} 187 \\ +\,\square\square\square \\ \hline 436 \end{array}$$

The boxes on the right are representing the missing addend. While this method does restate the subtraction problem in terms of an additive operation, the dynamics of the operation are not nearly as strong as in complementary numbers. For speed and accuracy, complementary numbers are to be preferred. However, missing addends should not be ignored and indeed they are very helpful when checking subtraction problems.

## HYPAI Converting (Mental Calculation)

HYPAI Converting is an optional lesson. It is intended for students already working with or capable of being introduced to negative numbers. It offers a convenient way of dealing with otherwise complex problems. It also reinforces the nature of subtraction and allows direct answers without the need of using HYPAI trades. Best of all it provides an excellent opportunity to practice with negative numbers.

The HYPAI Converting Method of subtraction should be considered optional. It should only be taught to students who are capable of dealing with negative numbers. Its chief feature is the means it provides for dealing with multiple subtrahends quickly and effectively.

Consider the following four problems:

| | | | |
|---|---|---|---|
| 6,000 | 100 | 20 | 4 |
| -1,000 | -800 | -60 | -8 |
| -2,000 | -700 | -30 | -5 |
| 3,000 | -1,400 | -70 | -9 |

Looking at the minuend and subtrahends only, the above problems can be rewritten into one problem, as they are below.

$$\begin{array}{r} 6{,}124 \\ -1{,}868 \\ -2{,}735 \\ \hline \end{array}$$

When we attempt to reach an answer, we are frustrated by the fact that the values in the units', tens', and hundreds' columns are all negative. The value of the answer in the thousands' column is positive. HYPAI Converting shows us how to reconcile the numbers.

There are three cases to be considered: 1) when an overall positive number is expected; 2) when an overall negative number is expected; and 3) when it is not clear whether our final answer will be positive or negative.

## (Case 1) Overall Result Expecting Positive Answer

By examining the thousands' column in the problem below, it is clear that the answer will be positive. When we reach a negative value in any one column, we must rename that number.

$$\begin{array}{r} 6,\ 1\ 2\ 4 \\ -1,\ 8\ 6\ 8 \\ \underline{-2,\ 7\ 3\ 5} \end{array}$$

When we calculate the value of the units' column, we find a result of negative nine. We cannot leave this as an answer but must restate it. However, if we add and then subtract the number ten to this number, we know we will not affect the value. By regrouping, however, we will change its name.

Thus $-9 = -9+(10-10) = (-9+10)-10$

$$= -10+(-9+10) = -10+1 = \begin{pmatrix} -10 \\ +\ 1. \end{pmatrix}$$

We now have values we can deal with. We place a +1 below the line in the units' column and place a -1 above the tens' column as a carry-over to reflect the -10. Our problem now looks like this.

$$\begin{array}{r} \phantom{6,\ 1\ }{}^{-1}\phantom{\ 4} \\ 6,\ 1\ 2\ 4 \\ -1,\ 8\ 6\ 8 \\ \underline{-2,\ 7\ 3\ 5} \\ 1 \end{array}$$

We must now approach the tens' column. Evaluating the tens, we reach an answer of -8. Again we rename in the following fashion.

$$-8 = -8+(+10-10)$$

$$= -10+(-8+10) = \begin{pmatrix} -10 \\ +\ 2 \end{pmatrix}$$

and the problem now appears like this.

$$\begin{array}{r} \phantom{6,\ 1\ }{}^{-1}\ {}^{-1}\phantom{\ 4} \\ 6,\ 1\ 2\ 4 \\ -1,\ 8\ 6\ 8 \\ \underline{-2,\ 7\ 3\ 5} \\ 2\ 1 \end{array}$$

Evaluating the hundreds' column, our original answer is -15. Once more, we rename as follows.

$$-15 = -15+(20-20) = (-15+20)-20$$

$$= -20+(-15+20) = -20+5 = \begin{pmatrix} -20 \\ +\ 5 \end{pmatrix}$$

We therefore place a +5 under the line in the hundreds' column and add a negative two into the thousands' column as a carry-over. Rewriting the problem as:

$$\begin{array}{r} {}^{-2}\ {}^{-1}\ {}^{-1}\phantom{\ 4} \\ 6,\ 1\ 2\ 4 \\ -1,\ 8\ 6\ 8 \\ \underline{-2,\ 7\ 3\ 5} \\ 5\ 2\ 1 \end{array}$$

Evaluating the thousands' column, we reach an answer of +1 which we are able to place in the answer directly.

$$\begin{array}{r} {}^{-2}\ {}^{-1}\ {}^{-1}\phantom{\ 4} \\ 6,\ 1\ 2\ 4 \\ -1,\ 8\ 6\ 8 \\ \underline{-2,\ 7\ 3\ 5} \\ 1,\ 5\ 2\ 1 \end{array}$$

You should complete the following problems before proceeding to Case 2.

# PRACTICE EXERCISES

## Case 1

| □ □ -1 -1 | □ □ □ □ |
|---|---|
| 72,403 | 43,215 |
| − 18,956 | − 28,697 |
| 47 | |

| □ □ -1 0 | □ □ □ □ |
|---|---|
| 65,218 | 70,136 |
| − 37,890 | − 27,543 |
| 28 | |

| □ □ 0 0 | □ □ □ □ |
|---|---|
| 72,067 | 32,184 |
| − 54,815 | − 17,652 |
| 52 | |

| □ □ -1 0 | □ □ □ □ |
|---|---|
| 52,807 | 60,739 |
| − 34,254 | − 27,561 |
| 53 | |

| □ □ □ □ | □ □ □ □ |
|---|---|
| 80,134 | 86,075 |
| − 59,276 | − 29,315 |

| □ □ □ □ | □ □ □ □ |
|---|---|
| 56,439 | 84,936 |
| − 18,764 | − 58,785 |

| □ 0 -1 | □ □ □ |
|---|---|
| 7,193 | 8,271 |
| − 3,604 | − 2,742 |
| − 925 | − 316 |
| 64 | |

| □ 0 -1 | □ □ □ |
|---|---|
| 5,326 | 7,183 |
| − 718 | − 4,564 |
| − 2,908 | − 219 |
| 00 | |

| □ 0 -1 | □ □ □ |
|---|---|
| 9,784 | 5,642 |
| − 613 | − 2,313 |
| − 1,429 | − 406 |
| 42 | |

| □ 0 -2 | □ □ □ |
|---|---|
| 6,842 | 7,568 |
| − 3,804 | − 937 |
| − 929 | − 4,515 |
| 09 | |

| □ □ □ | □ □ □ |
|---|---|
| 6,082 | 8,051 |
| − 134 | − 426 |
| − 4,509 | − 3,716 |

| □ □ □ | □ □ □ |
|---|---|
| 4,268 | 6,285 |
| − 539 | − 746 |
| − 3,729 | − 2,119 |

| □ -2 0 | □ □ □ |
|---|---|
| 9,435 | 7,869 |
| − 692 | − 2,954 |
| − 3,473 | − 635 |
| 70 | |

| □ -1 0 | □ □ □ |
|---|---|
| − 241 | − 13 |
| − 74 | − 492 |
| 5,139 | 6,078 |
| 24 | |

| □ -2 0 | □ □ □ |
|---|---|
| − 692 | − 182 |
| 7,648 | 4,617 |
| − 574 | − 790 |
| 82 | |

| □ -2 0 | □ □ □ |
|---|---|
| 6,538 | 9,015 |
| − 1,382 | − 4,280 |
| − 2,796 | − 2,534 |
| 60 | |

□□□
3,787
− 251
− 1,846

□□□
− 421
8,306
− 594

□□□
9,120
− 6,285
− 65

□□□
− 80
− 465
7,312

□□□
− 432
− 10
2,306

□□□
7,026
− 1,823
− 3,290

□□□
− 1,693
7,206
− 67

□□□
7,317
− 865
− 94

□-2-1
6,124
− 869
− 2,785
70

□□□
7,037
− 189
− 98

□-1+1
7,068
6,129
− 1,853
− 2,670
74

□□□
5,208
− 8,661
9,047
− 1,932

□-1-2
− 687
5,961
− 735
39

□□□
− 3,609
9,014
− 865

□0-2
9,180
− 3,429
− 1,539
8,017
29

□□□
− 1,538
− 6,468
9,074
7,192

□-1-2
− 69
− 926
7,010
15

□□□
− 796
− 90
1,635

□-2 0
− 1,382
7,804
6,911
− 5,263
70

□□□
6,527
− 4,962
− 1,835
9,204

□-2-1
6,743
− 895
− 1,687
61

□□□
7,823
− 997
− 56

□□□
− 2,851
7,439
− 3,982
9,128

□□□
7,281
− 5,967
− 3,848
5,364

## (Case 2) Overall Result Expecting Negative Answer.

Occasionally you will encounter problems where the overall result is negative, but the individual columns reach a positive value such as in the problem below.

```
  3, 9 7 8
- 2, 1 0 5
- 2, 0 2 2
```

Once again, the numbers must be converted. Examining the thousands' column, we can see that the overall answer is going to be negative. When we calculate the units' column, we get a result of +1.

$$+1 = (+1-10)+10$$

$$= -9+10 = \begin{pmatrix} +10 \\ -\ 9. \end{pmatrix}$$

We write the 9 below the line and place a one in the tens' column for the +10.

```
    +1
  3, 9 7 8
- 2, 1 0 5
- 2, 0 2 2
-------------
        9
```

When we turn to the tens' column, our preliminary result is +6. However

$$+6 = (+6-10)+10$$

$$= -4+10 = \begin{pmatrix} +10 \\ -\ 4. \end{pmatrix}$$

We thus place a 4 below the line in the tens' column and a +1 in the hundreds' column.

```
    +1 +1
  3, 9 7 8
- 2, 1 0 5
- 2, 0 2 2
-------------
        4 9
```

In calculating the hundreds' column we must convert +9.

$$+9 = (+9-10)+10$$

$$= -1+10 = \begin{pmatrix} +10 \\ -\ 1. \end{pmatrix}$$

Our problem now appears as below.

```
    +1 +1 +1
  3, 9 7 8
- 2, 1 0 5
- 2, 0 2 2
-------------
    - 1 4 9
```

When calculating the thousands' column we find an answer of 0. The only thing left to do is to place a minus sign in the answer. (Remember we were changing to negative values.) The answer thus becomes -149.

## EXERCISES

### Case 2

```
 □ 0 0          □ □ □
  4, 3 7 3       1, 8 0 6
  3, 1 2 0       3, 3 1 5
- 9, 0 9 6     - 7, 2 9 7
-------------  -----------
-      0 3     -
```

```
 □ +1 +2        □ □ □
  1, 3 8 9       4, 4 0 8
- 5, 7 6 4     - 7, 2 3 1
  2, 0 3 6       2, 0 5 7
-------------  -----------
-      3 9     -
```

```
 □ +1 0         □ □ □
- 8, 0 6 5     - 9, 7 4 5
  1, 9 7 3       4, 1 8 4
  4, 8 9 2       1, 3 6 1
-------------  -----------
-      0 0     -
```

Once the method of dealing with the various numbers used to convert is understood, teachers can provide a valuable lesson for their students by having the students draw up conversion charts such as those below. The converted numbers are shown in columns to emphasize the place values.

## HYPAI CONVERSION CHART

### FOR CASE 1—Overall Result Expecting Positive Answers

Whenever negative answer is produced, it should be converted on that column as follows.

| Column Answer | Conversion | Carry-over to Next Column | Answer for that Column |
|---|---|---|---|
| 0 = | 0<br>+ 0 | 0 | +0 |
| -1 = | -10<br>+ 9 | -1 | +9 |
| -2 = | -10<br>+ 8 | -1 | +8 |
| -3 = | -10<br>+ 7 | -1 | +7 |
| -4 = | -10<br>+ 6 | -1 | +6 |
| -5 = | -10<br>+ 5 | -1 | +5 |
| -6 = | -10<br>+ 4 | -1 | +4 |
| -7 = | -10<br>+ 3 | -1 | +3 |
| -8 = | -10<br>+ 2 | -1 | +2 |
| -9 = | -10<br>+ 1 | -1 | +1 |
| -10 = | -10<br>+ 0 | -1 | +0 |

### FOR CASE 2—Overall Result Expecting Negative Answers

Whenever positive answer is produced, it should be converted on that column as follows.

| Column Answer | Conversion | Carry-over to Next Column | Answer for that Column |
|---|---|---|---|
| 0 = | 0<br>- 0 | 0 | -0 |
| +1 = | +10<br>- 9 | +1 | -9 |
| +2 = | +10<br>- 8 | +1 | -8 |
| +3 = | +10<br>- 7 | +1 | -7 |
| +4 = | +10<br>- 6 | +1 | -6 |
| +5 = | +10<br>- 5 | +1 | -5 |
| +6 = | +10<br>- 4 | +1 | -4 |
| +7 = | +10<br>- 3 | +1 | -3 |
| +8 = | +10<br>- 2 | +1 | -2 |
| +9 = | +10<br>- 1 | +1 | -1 |
| +10 = | +10<br>- 0 | +1 | -0 |

| FOR CASE 1—Overall Result Expecting Positive Answers<br><br>Whenever negative answer is produced, it should be converted on that column as follows. | | | | FOR CASE 2—Overall Result Expecting Negative Answers<br><br>Whenever positive answer is produced, it should be converted on that column as follows. | | | |
|---|---|---|---|---|---|---|---|
| **Column Answer** | **Conversion** | **Carry-over to Next Column** | **Answer for that Column** | **Column Answer** | **Conversion** | **Carry-over to Next Column** | **Answer for that Column** |
| -11 = | -20<br>+ 9 | -2 | +9 | +11 = | +20<br>- 9 | +2 | -9 |
| -12 = | -20<br>+ 8 | -2 | +8 | +12 = | +20<br>- 8 | +2 | -8 |
| -13 = | -20<br>+ 7 | -2 | +7 | +13 = | +20<br>- 7 | +2 | -7 |
| -14 = | -20<br>+ 6 | -2 | +6 | +14 = | +20<br>- 6 | +2 | -6 |
| -15 = | -20<br>+ 5 | -2 | +5 | +15 = | +20<br>- 5 | +2 | -5 |
| -16 = | -20<br>+ 4 | -2 | +4 | +16 = | +20<br>- 4 | +2 | -4 |
| -17 = | -20<br>+ 3 | -2 | +3 | +17 = | +20<br>- 3 | +2 | -3 |
| -18 = | -20<br>+ 2 | -2 | +2 | +18 = | +20<br>- 2 | +2 | -2 |
| -19 = | -20<br>+ 1 | -2 | +1 | +19 = | +20<br>- 1 | +2 | -1 |
| -20 = | -20<br>+ 0 | -2 | +0 | +20 = | +20<br>- 0 | +2 | -0 |
| Etc. | | | | Etc. | | | |
| Remarks: Whenever positive answer is produced, it is direct answer for that column. | | | | Remarks: Whenever negative answer is produced, it is direct answer for that column. | | | |

## (CASE 3) Overall Result Presuming Positive Answer

Sometimes there are examples for which we do not know whether the overall result will be positive or negative. Consider the problem:

```
      -2 -1
       5, 4 3 1
       7, 1 4 5
    - 2, 8 9 2
    - 9, 9 6 0
    ----------
      -1,+7 +2 +4
```

Initially we proceed on the presumption the answer will be positive. We calculate the units' column and find a +4. We write the +4 below the line and proceed to the tens' column. Calculating the tens' column, the preliminary result is -8 which we convert to:

```
      -10
+2-10 = + 2.
```

We place the + 2 below the line in the tens' column and a -1 in the hundreds' column for a carry-over. Evaluating the hundreds' column, we find a -13 which must be converted to:

```
      -20
+7-20 = + 7.
```

We place the +7 below the line in the hundreds' column and place a -2 in the thousands' column for a carry-over.

When we calculate the thousands' column, we recognize that we still have not completed our problem. The value of the thousands' column is negative. All the other numbers are positive. We now have -1,000 and +724 shown in the example.

```
       5, 4 3 1
       7, 1 4 5
    - 2, 8 9 2
    - 9, 9 6 0
    ----------
    - 1, 0 0 0
         +7 +2 +4
```

The positive numbers must be reconverted, just as they were in Case 2, in the manner of overall result expecting negative answer.

## FOR RECONVERSION INTO A NEGATIVE ANSWER

```
      +1 +1 +1
    - 1, 0 0 0
        + 7 2 4
    ----------
    - 0, 2 7 6
```

Thus in the units' the + 4 = +10→+1 is carry-over for tens' column
- 6→Answer for the units' column

in the tens' the +2 +1 from the carry-over = +10→+1 is carry-over for hundreds' column
- 7→answer for the tens' column

in the hundreds' the +7 +1 from the carry-over = +10→+1 is carry-over for thousands' column
- 2→answer for the hundreds' column

in the thousands' the -1 +1 from the carry-over = 0

Therefore our answer is -276.

## REVIEW OF THE SUBTRACTION METHODS

You have been presented with six methods that can be used when performing subtraction problems. The purpose of this chapter is to review the features of the six methods and in particular to point out the various strengths of each method.

### Clearing One-by-One

When young students are first introduced to subtraction, they should be invited to press the units of minuend, one-by-one and then to clear the subtrahend, one-by-one. This is the type of subtraction which you saw in Stage 1 Part A. The reason why so much space was devoted to this methodology was because the author wanted the students to physically feel how subtraction is the exact reverse of addition. When the students see how all of the manipulations of additon have their counterpart in subtraction; and when they begin to feel how subtraction always involves decreasing fingers, the young students can gain a great insight into the nature of subtraction.

## HYPAI TRADING

HYPAI Trading was the next method to be introduced. This method follows conventional mathematics insofar as a trade is only made into a column that requires it. Thus the students are able to coordinate the Trading method with their conventional books. The difference between HYPAI Trading and conventional mathematics is that HYPAI Trading is much more explicit and shows precisely what is happening in each column.

## CONSISTENT HYPAI TRADING

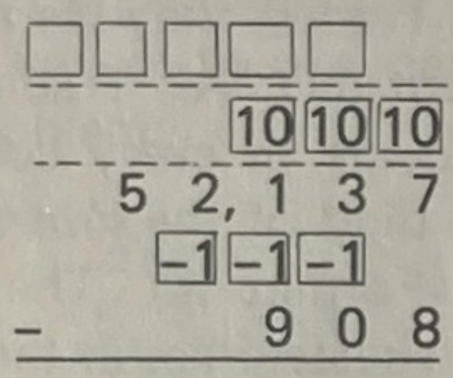
Once the students were comfortable with the HYPAI Trades, the/ were introduced to Consistent HYPAI Trading where a trade was made in every column where there was a digit of the *subtrahend.* (See iliustration Format A). The over-riding merit of this system lies in the fact that it eliminates from the students' minds the worry over whether or not they have to regroup. Instructing them to make a trade for every digit in the subtrahend relieves them of the necessity of the decision making. When they reached mixed operations and multiple subtrahends, the trading numbers were moved to the top of the problem and a trade was made for every digit in the *minuend.* (See illustration Format B-1, B-2).

## HYPAI POWER TRADING

In order to increase speed and in order to accommodate multiple subtrahends and subtrahends found in unusual places, e.g. the first entry in a problem, the trading numbers were all moved to the top of the problem and a new format, HYPAI Power Trading was presented. Here the instruction was to make a trade for every digit in the *minuend.* (See illustration Format C-1.) The author feels that Consistent HYPAI Trading and HYPAI Power Trading are very closely related and it is with these two methods that teachers should concentrate in order to teach their students what is happening when they are regrouping. Both of these methods are quickly absorbed and lead to very rapid calculation. The students welcome the consistent approach and soon grow to feel very comfortable with the topic of regrouping.

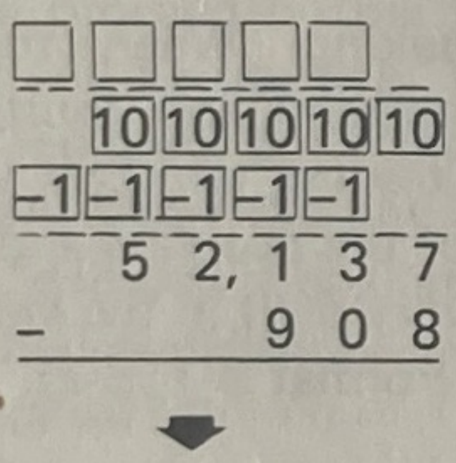
## CONSISTENT FORMAT (A)

Make a trade for every column of the subtrahend.



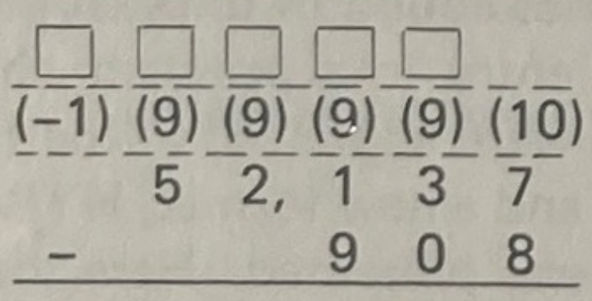
## CONSISTENT FORMAT (B-1)

Make a trade for every column of the minuend.



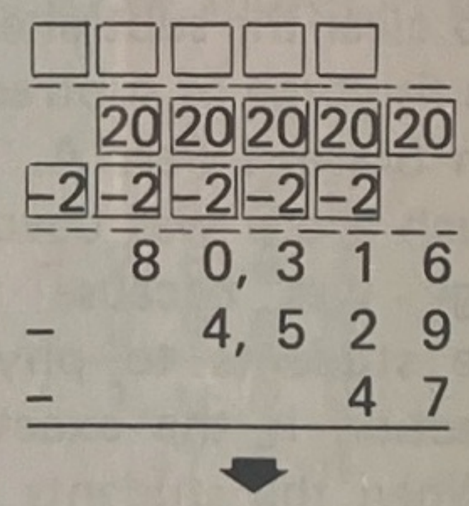
## POWER FORMAT (C-1)

Combine the trading numbers.



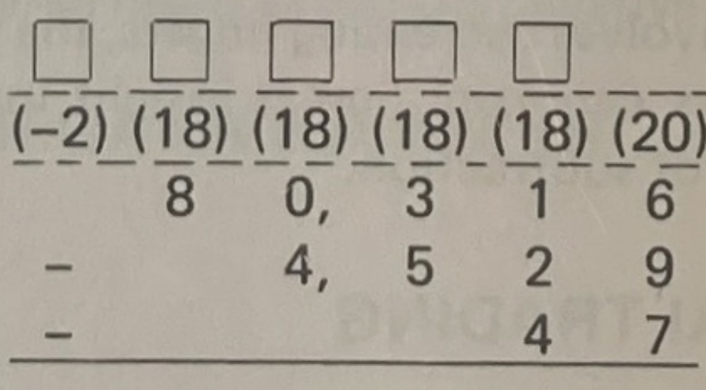
## CONSISTENT FORMAT (B-2)

Make a trade for every column of the minuend.



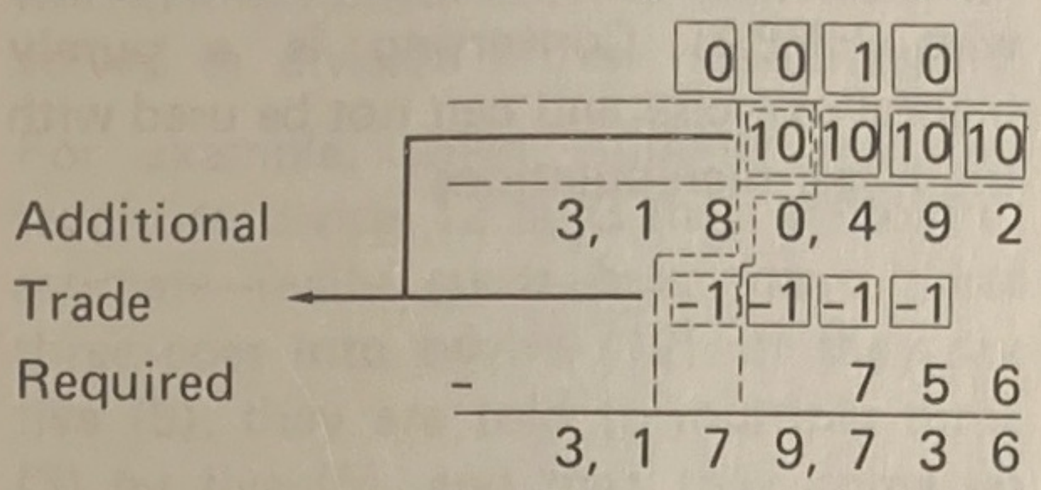
## POWER FORMAT (C-2)

Combine the trading numbers.

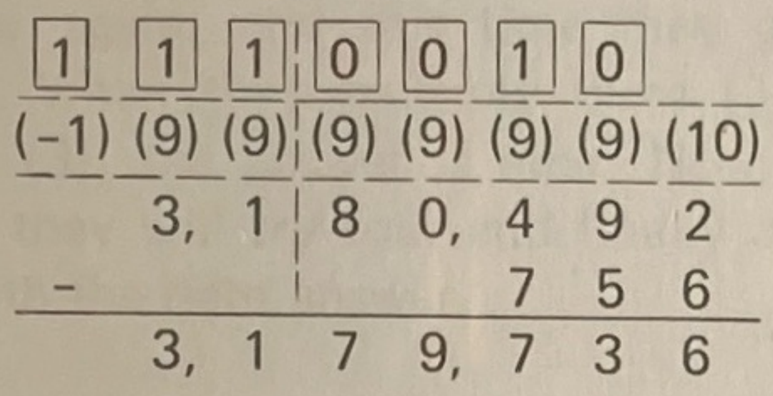


The first format of Consistent HYPAI Trading and HYPAI Power Trading was designed for problems that occur over 90% of the time and provided rules for rapid calculation. However, there is a special group of problems. The problem 3,180,492–756 requires an additional trade into the thousands' column when using the 1st format of Consistent HYPAI Trading since that format was based on the subtrahend. Initially, in HYPAI Power Trading the same problem required too much calculation.

**Consistent HYPAI Trading (Format (A)**




**HYPAI Power Trading**




However, the exceptions do not prove difficult to the students. When trading was first introduced, the students were taught to make a trade whenever it was required. Hence in this kind of special problem, the students are able to see that an extra trade into the thousands' column is required.

In HYPAI Power Trading, most students very quickly realize that once they have a 1 in a carry-over box *beyond the subtrahend,* all the remaining carry-over boxes will automatically contain 1's, and they stop calculating and just bring down the rest of the minuend.

One final point about these special cases. The author realized that they existed and gave numerous examples when HYPAI Trading was being taught of problems such as (80–1), (100–5), (306–8), (7,000–6). Thus many of the students are capable of calculating a problem such as the one shown above, using Consistent HYPAI Trading through the hundreds' column and then calculating the rest of the problem mentally.

## Complementary Numbers

Complementary Numbers were introduced as a method for rapid calculation of multi-digit numbers. It is quickest when there are only 1 or 2 subtrahends. It can also be used as a *last resort* for severely handicapped students, but the teacher should be aware that the student will only be finding out *how* to get an answer; *not why* the answer is true. It gives little insight into the nature of subtraction.

## HYPAI Converting (Mental Calculation)

HYPAI Converting can only be used with students who are capable of dealing with negative numbers. It is extremely useful when there is a long list of numbers to be calculated. After the initial demonstration of why the method works, students can use the HYPAI Conversion chart and transpose any single digit in the answer into a digit with the proper sign. If the students have the skill of dealing with negative numbers, they will find that adding the converting skill enables them to calculate much more rapidly.

## Mixed Operations

When you have problems with multiple subtrahends, or problems with both addends and subtrahends, you may use any of 4 methods to find your answer. Consistent HYPAI Trading, HYPAI Power Trading, Complementary Numbers, and HYPAI Converting may all be used for these complicated problems. They enable students to calculate these problems without having to rewrite the problem grouping all the addends and subtrahends. The author feels that when first teaching these types of problems greater insight will be achieved by using Consistent HYPAI Trading and HYPAI Power Trading. After the students understand precisely what is happening, they may be able to achieve greater speed in calculation if they use Complementary Numbers and HYPAI Converting.

## Relationship Between the Methods and the 4 Stages.

Clearing one-by-one is the definition of Stage 1 and thus can only be used at that level. HYPAI Trading, Consistent HYPAI Trading, HYPAI Power Trading and Complementary Numbers are adaptable to all 4 stages of Chisanbop ™ . Students should use the highest level manipulations that they are acquainted with. HYPAI Converting is a purely mental exercise and can not be used with the finger manipulations.

## DIVISION STAGE 1

In the conventional methodology, division is taught by using multiplication. While the conventional methodology gives lip service to the notions that multiplication is an extension of addition and division is an extension of subtraction, in practice, division is taught by using multiplication—in other words guessing.

When students have become sufficiently expert at multiplication, they can do fairly well at guessing the answer in division. But until they reach that degree of expertise in multiplication, the guesswork involved in division proves mind-boggling.

For example, when students are first taught to divide 12 by 3 they are told to estimate—really guess—how many times three goes into twelve (12). If they say five (5), they are told to multiply three (3) by five (5), and then they come up with fifteen (15). So then they are told to try again, and this time they guess three. When they multiply three (3) by three (3), the answer is nine. Now perhaps, they will try four and finally come up with the right answer.

In Chisanbop™, the students see right from the start that division is repetitive subtraction. When asked to divide 12 by 3, they proceed with sureness. They don't guess 5 or 3 as the answer, nor do they multiply. Instead, they solve the problem the way every division problem should be solved.

In Chisanbop™, we teach division as an extension of subtraction right from the start, so the student is always aware of the true nature of the dividing process.

A further advantage of teaching division as an extension of subtraction is that it makes it much easier for students to understand the concept of fractions. By using subtraction, it is quite obvious to students that the remainder—when there is one—must always be less than the divisor. If such is not the case, they immediately appreciate that they can deduct the divisor at least one more time.

In the Chisanbop™ method, young students use a HYPAI Counting Aid Chart in the beginning stage of division. (See following pages for chart and instructions.)

## HYPAI COUNTING AID CHART

Just as a Counting Aid Chart was helpful in Multiplication Stage 1 to give an insight into the nature of addition, a similar chart is necessary to understand the meaning of division.

Since division is really repetitive subtraction, the operation should be explained linearly. For instance, in the problem 28÷4, we should explain:

Twenty-eight minus one four is 24.

Twenty-eight minus two fours is 24 minus another four, 20.

Twenty-eight minus three fours is 20 minus another four, 16.

Twenty-eight minus four fours is 16 minus another four, 12.

Twenty-eight minus five fours is 12 minus another four, 8.

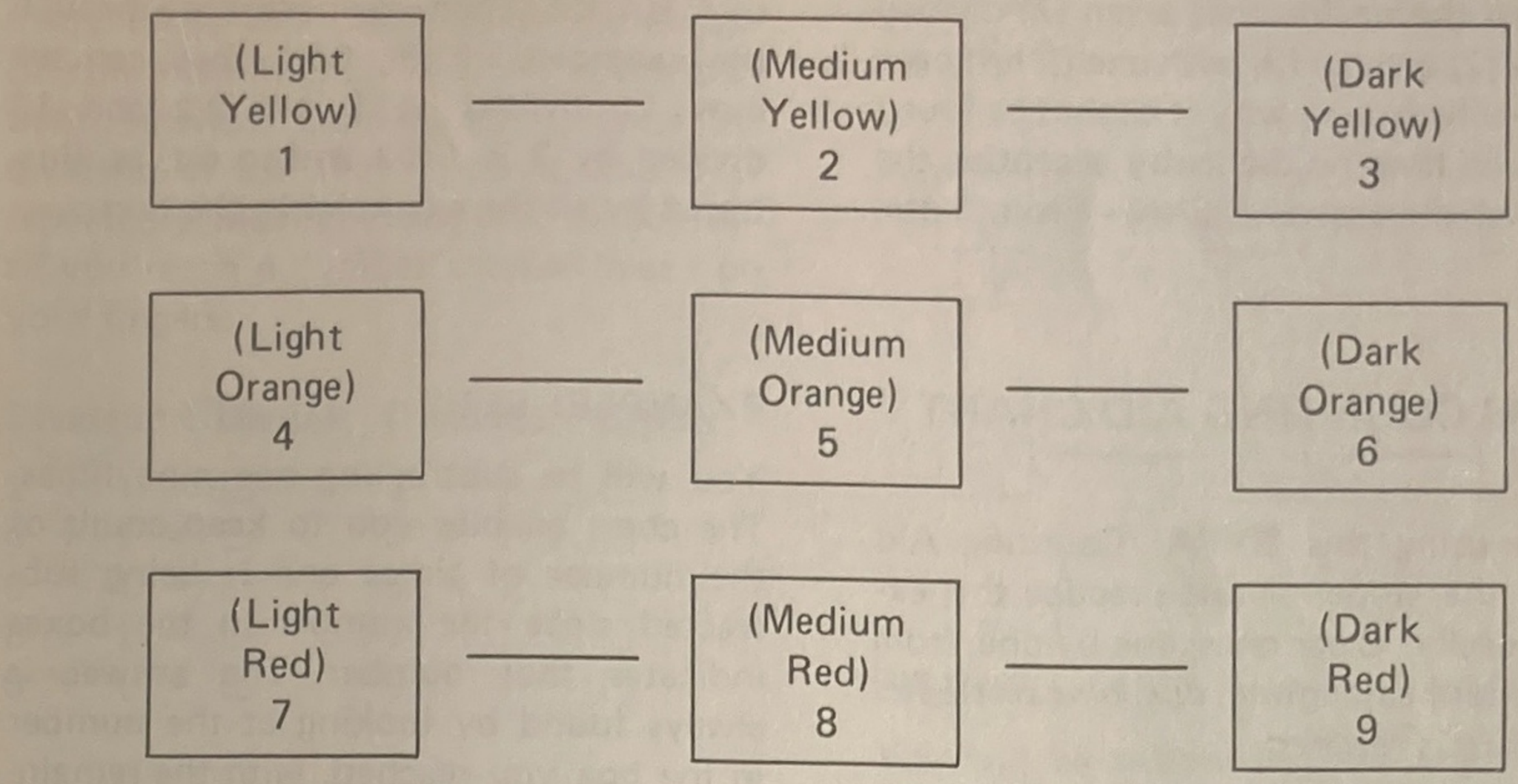
Twenty-eight minus six fours is 8 minus another four, 4.

Twenty-eight minus seven fours is 4 minus another four, 0.

There are four elements that students must master: a) the unit they are repeatedly subtracting b) how many times they have to repeatedly subtract that unit c) remembering the intermediate remainders as they are in the process d) the mental calculation of subtracting.

Using the HYPAI Counting Aid Chart, many of the initial problems of understanding are alleviated. The dividend (number being divided) is initially pressed on the fingers; how many times a number has been subtracted (cleared) is counted by which box the fingers are resting on; the intermediate remainders are registered on the fingers as they are progressively cleared; and finally the quotient, (answer) is found in the box where the fingers come to a final resting place.

Once again, the positions of the numbers in the box and the indicated colors are meant to impress the students with the changing of the quotient and to leave them with the clear understanding that when they were asked to divide a number by a divisor, they were really asked to count the number of times that divisor could be subtracted from (cleared from) the dividend. As in multiplication, use of this chart will give a quick insight into the nature of division.




The chart enables students to keep track of how many times a number is being subtracted. Before using the HYPAI Counting Aid Chart, the student must practice these exercises orally.

Clear ones from ten—saying *one, one, one,* etc. until zero is reached.

Clear twos from twenty—saying *one two, one two, one two,* until zero is reached.

Clear threes from thirty—saying *one two three, one two three,* etc. until zero is reached.

Clear fours from forty—saying *one two three four, one two three four,* etc. until zero is reached.

Clear fives from fifty—saying *one two three four five, one two three four five,* etc. until zero is reached.

The answer is never established until the students are asked to read the number shown in the box of the HYPAI Counting Aid Chart when they cannot subtract the number any more. (When they still have fingers pressed but they cannot subtract the divisor one more time, the pressed fingers are the remainder.) Through the use of the HYPAI Counting Aid Chart the student will inevitably understand simple division.

After the students thoroughly understand how to divide when the answer never involves a remainder, they can be given problems where there will be remainders. They will not cause them any difficulty if they are thoroughly grounded in the fundamentals of division through the use of the HYPAI Counting Aid Chart.

Thus students who are told to divide seven (7) by two (2) will see very readily that,

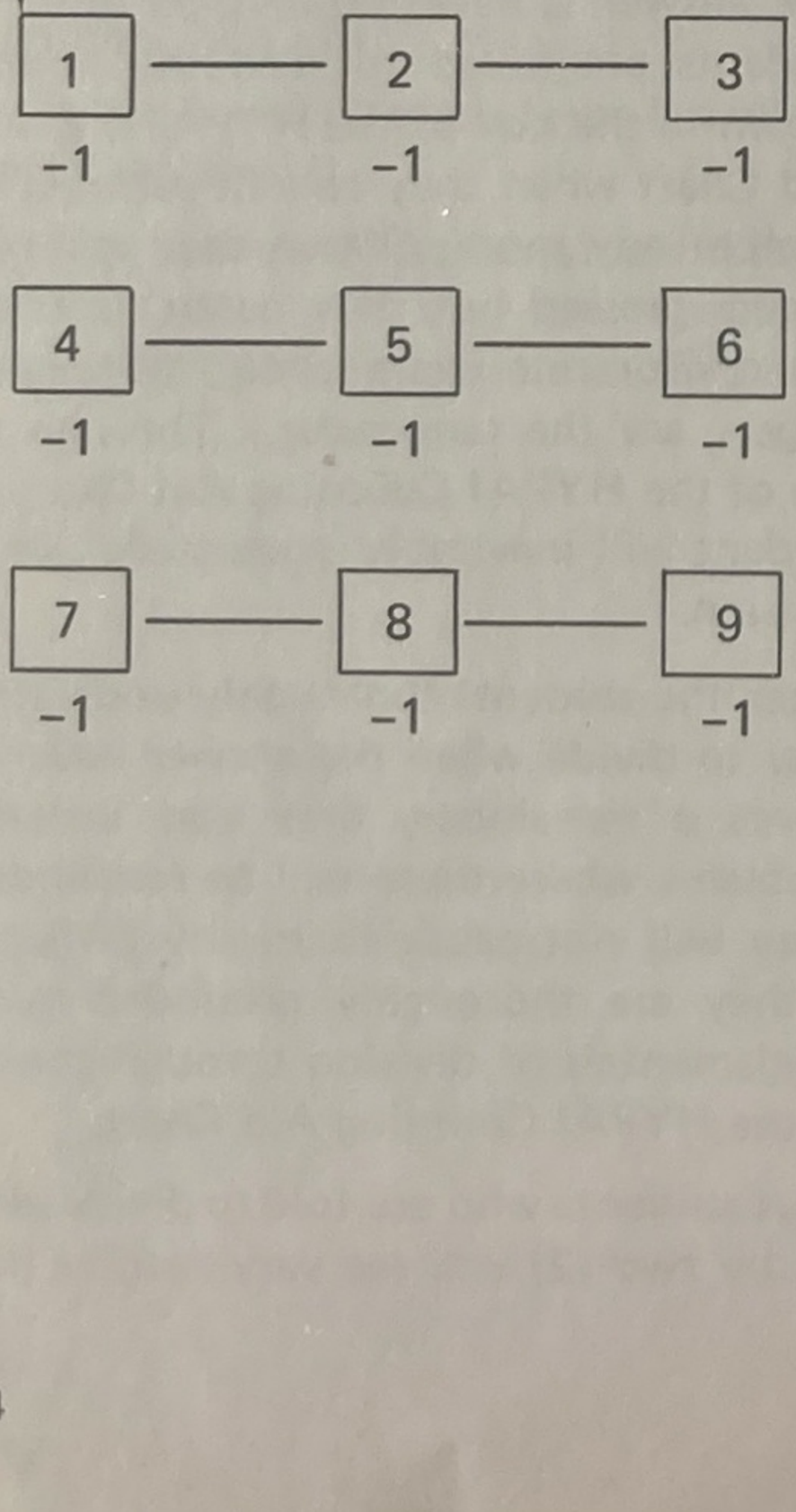
after they have taken two (2) from seven (7) three times, they cannot take it a fourth time. It will be very easy for them to grasp the notion that seven (7) divided by two (2) is three (3) with one (1) left over. (They will also see why it cannot be four.) They will have no difficulty accepting the fact that the answer is 3 1/2. From there, it will be simple (if they work through all the examples provided) to understand how 5 divided by 2 is 2 1/2 and 3 divided by 2 is 1 1/2. Then, again working through the examples of the text, they can see how 11 divided by 3 is 3 2/3 and 13 divided by 3 is 4 1/3 and so on, as illustrated by all the examples in the text.

## HYPAI COUNTING AID CHART

### ONES

Before using this HYPAI Counting Aid Chart, the student must practice this exercise orally. Clear ones, one-by-one, from onety (ten) saying *one, one, one, one*...etc. until zero is reached.

### DIVIDEND (The Number Being Divided) STARTING POINT




EXAMPLE: 9÷1

You will be subtracting one nine times. The chart enables you to keep count of the number of times one is being subtracted since the number in the boxes indicates that number. The answer is always found by looking at the number in the box you reached, with the remainder showing on the fingers.

*Procedure:*

Move your fingers from square to square as you say, "clear *one...one, one, one,...* etc."

When you reach square 9, read the number in the box where you stopped. After this method is practiced many times, you will be able to move only your eyes from square to square as you manipulate your fingers. Ultimately, you are able to keep track mentally of how many times you have subtracted a number.

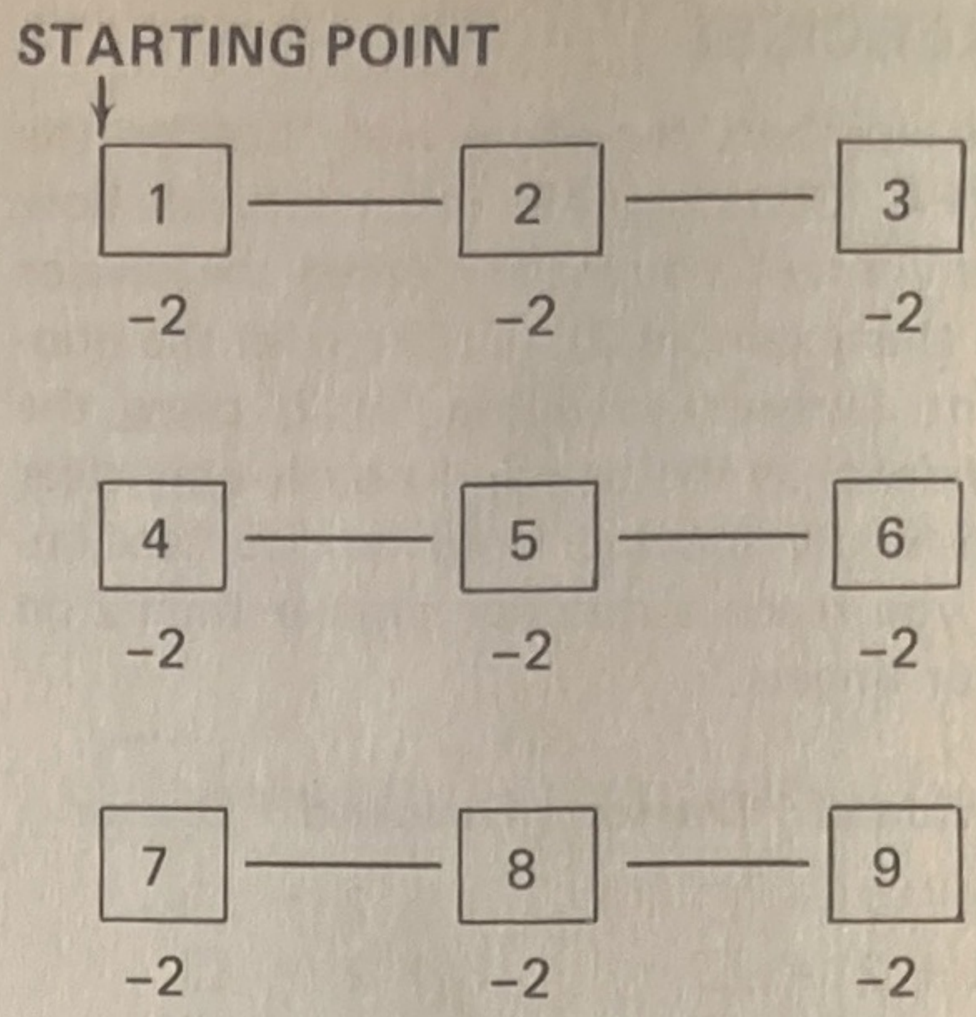
## EXERCISES

The numbers shown in each box on the HYPAI Counting Aid Chart indicate how many times you have cleared the divisor (in this example 1). To arrive at the quotient (answer) dividing by 1, press the dividend at the starting point, and clear repeatedly moving from box to box until you reach a number smaller than 1 on your fingers.

| Dividend ÷ Divisor | Dividend ÷ Divisor |
|---|---|
| 2 ÷ 1 = □ | 1 ÷ 1 = □ |
| 4 ÷ 1 = □ | 3 ÷ 1 = □ |
| 6 ÷ 1 = □ | 5 ÷ 1 = □ |
| 8 ÷ 1 = □ | 7 ÷ 1 = □ |
| 0 ÷ 1 = □ | 9 ÷ 1 = □ |
| 7 ÷ 1 = □ | 6 ÷ 1 = □ |
| 5 ÷ 1 = □ | 4 ÷ 1 = □ |
| 9 ÷ 1 = □ | 0 ÷ 1 = □ |
| 1 ÷ 1 = □ | 2 ÷ 1 = □ |
| 3 ÷ 1 = □ | 8 ÷ 1 = □ |
| 3 ÷ 1 = □ | 4 ÷ 1 = □ |
| 6 ÷ 1 = □ | 0 ÷ 1 = □ |
| 8 ÷ 1 = □ | 1 ÷ 1 = □ |
| 5 ÷ 1 = □ | 7 ÷ 1 = □ |
| 9 ÷ 1 = □ | 2 ÷ 1 = □ |

### TWOS

Before using this HYPAI Counting Aid Chart, the student must practice this exercise orally. Clear twos, one-by-one, from twenty saying *one two, one two,* etc. until zero is reached.




EXAMPLE: 16÷2

You will be subtracting two eight times. The chart enables you to keep count of the number of times two is being subtracted since the number in the boxes indicates that number. The answer is always found by looking at the number in the box you reached, with the remainder showing on the fingers.

*Procedure:*

Move your fingers from square to square as you say, "Clear two...*one two, one two, one two,*...etc." When you reach square 8, read the number in the box where you stopped. After this method is practiced many times, you will be able to move only your eyes from square to square as you manipulate your fingers. Ultimately, you are able to keep track mentally of how many times you have subtracted a number.

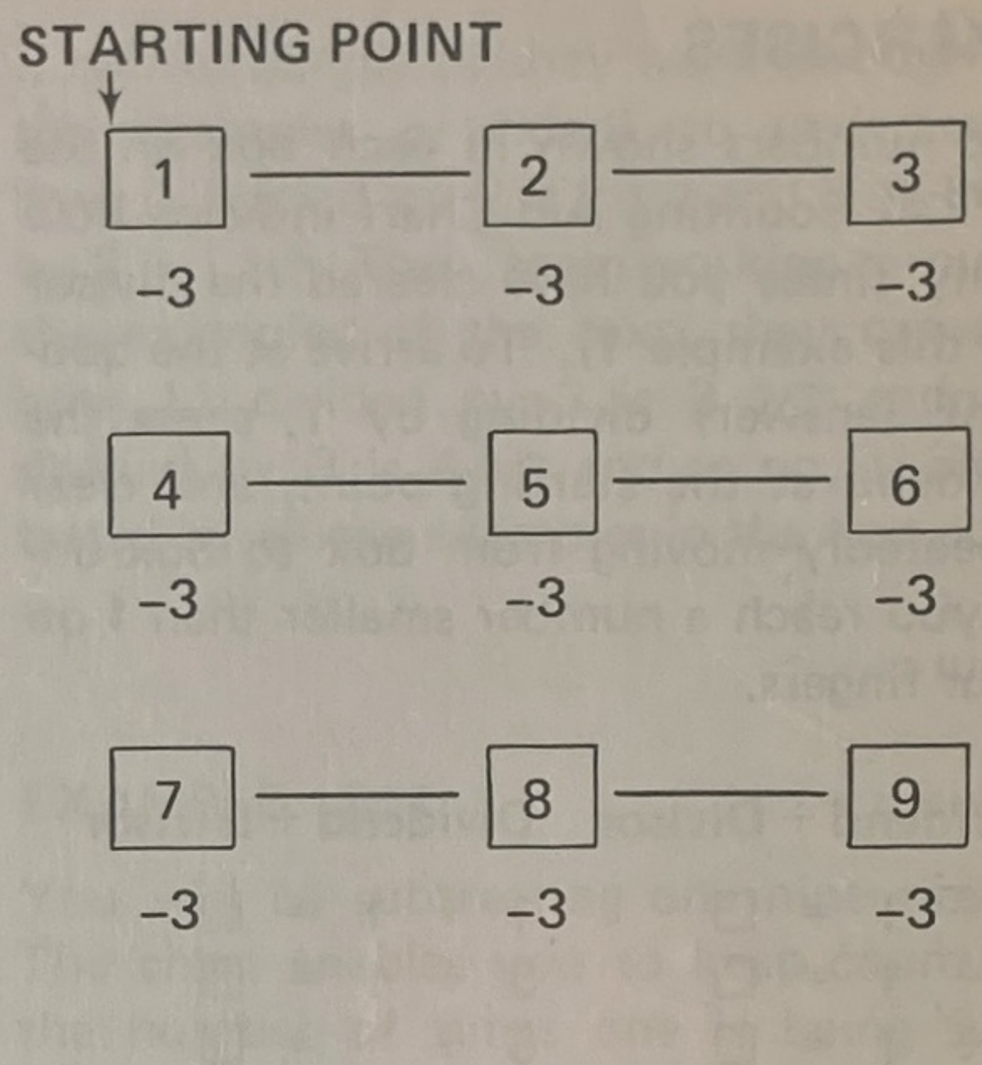
## EXERCISES

The numbers shown in each box on the HYPAI Counting Aid Chart indicate how many times you have cleared the divisor (in this example 2). To arrive at the quotient (answer) dividing by 2, press the dividend at the starting point, and clear repeatedly moving from box to box until you reach a number smaller than 2 on your fingers.

| Dividend ÷ Divisor | Dividend ÷ Divisor |
|---|---|
| 2 ÷ 2 = □ | 4 ÷ 2 = □ |
| 6 ÷ 2 = □ | 8 ÷ 2 = □ |
| 10 ÷ 2 = □ | 12 ÷ 2 = □ |
| 14 ÷ 2 = □ | 16 ÷ 2 = □ |
| 18 ÷ 2 = □ | 0 ÷ 2 = □ |
| 12 ÷ 2 = □ | 14 ÷ 2 = □ |
| 8 ÷ 2 = □ | 10 ÷ 2 = □ |
| 0 ÷ 2 = □ | 18 ÷ 2 = □ |
| 4 ÷ 2 = □ | 2 ÷ 2 = □ |
| 16 ÷ 2 = □ | 6 ÷ 2 = □ |
| 5 ÷ 2 = □ | 3 ÷ 2 = □ |
| 9 ÷ 2 = □ | 7 ÷ 2 = □ |
| 13 ÷ 2 = □ | 11 ÷ 2 = □ |
| 17 ÷ 2 = □ | 15 ÷ 2 = □ |
| 1 ÷ 2 = □ | 19 ÷ 2 = □ |

## THREES

Before using this HYPAI Counting Aid Chart, the student must practice this exercise orally. Clear threes, one-by-one, from thirty saying *one, two, three, one, two, three,* etc. until zero is reached.




EXAMPLE: 21÷3

You will be subtracting three seven times. The chart enables you to keep count of the number of times three is being subtracted since the number in the boxes indicates that number. The answer is always found by looking at the number in the box you reached, with the remainder showing on the fingers.

*Procedure:*

Move your fingers from square to square as you say, "Clear three...*one two three, one two three, one two three,*...etc." When you reach square 7, read the number in the box where you stopped. After this method is practiced many times, you will be able to move only your eyes from square to square as you manipulate your fingers. Ultimately, you are able to keep track mentally of how many times you have subtracted a number.

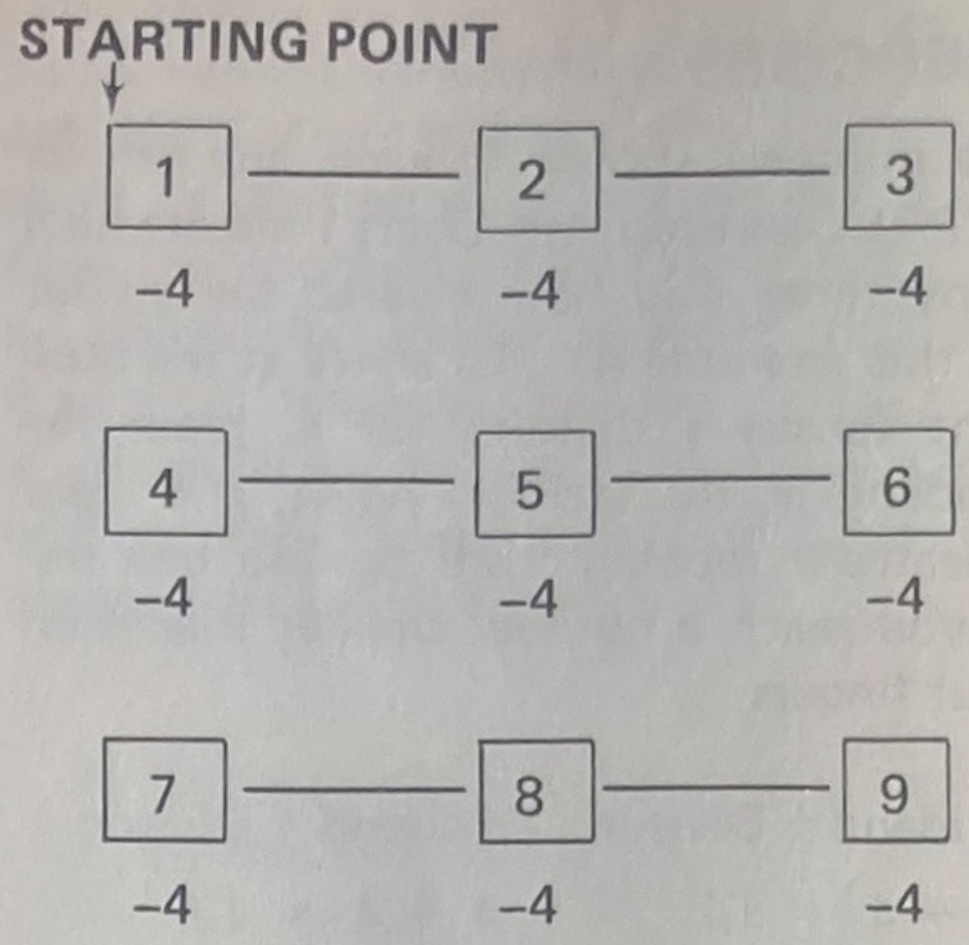
## EXERCISES

The numbers shown in each box on the HYPAI Counting Aid Chart indicate how many times you have cleared the divisor (in this example 3). To arrive at the quotient (answer) dividing by 3, press the dividend at the starting point, and clear repeatedly moving from box to box until you reach a number smaller than 3 on your fingers.

| Dividend ÷ Divisor | Dividend ÷ Divisor |
|---|---|
| 6 ÷ 3 = ☐ | 3 ÷ 3 = ☐ |
| 12 ÷ 3 = ☐ | 9 ÷ 3 = ☐ |
| 18 ÷ 3 = ☐ | 15 ÷ 3 = ☐ |
| 24 ÷ 3 = ☐ | 21 ÷ 3 = ☐ |
| 0 ÷ 3 = ☐ | 27 ÷ 3 = ☐ |
| 21 ÷ 3 = ☐ | 18 ÷ 3 = ☐ |
| 15 ÷ 3 = ☐ | 12 ÷ 3 = ☐ |
| 27 ÷ 3 = ☐ | 0 ÷ 3 = ☐ |
| 3 ÷ 3 = ☐ | 6 ÷ 3 = ☐ |
| 9 ÷ 3 = ☐ | 24 ÷ 3 = ☐ |
| 4 ÷ 3 = ☐ | 7 ÷ 3 = ☐ |
| 10 ÷ 3 = ☐ | 22 ÷ 3 = ☐ |
| 16 ÷ 3 = ☐ | 19 ÷ 3 = ☐ |
| 22 ÷ 3 = ☐ | 25 ÷ 3 = ☐ |
| 25 ÷ 3 = ☐ | 1 ÷ 3 = ☐ |

## FOURS

Before using this HYPAI Counting Aid Chart, the student must practice this exercise orally. Clear fours, one-by-one, from forty saying *one two three four, one two three four,* etc. until zero is reached.




EXAMPLE: 24÷4

You will be subtracting four six times. The chart enables you to keep count of the number of times four is being subtracted since the number in the boxes indicates that number. The answer is always found by looking at the number in the box you reached, with the remainder showing on the fingers.

*Procedure:*

Move your fingers from square to square as you say, "Clear four...*one two three four, one two three four,*...etc." When you reach square 6, read the number in the box where you stopped. After this method is practiced many times, you will be able to move only your eyes from square to square as you manipulate your fingers. Ultimately, you are able to keep track mentally of how many times you have subtracted a number.

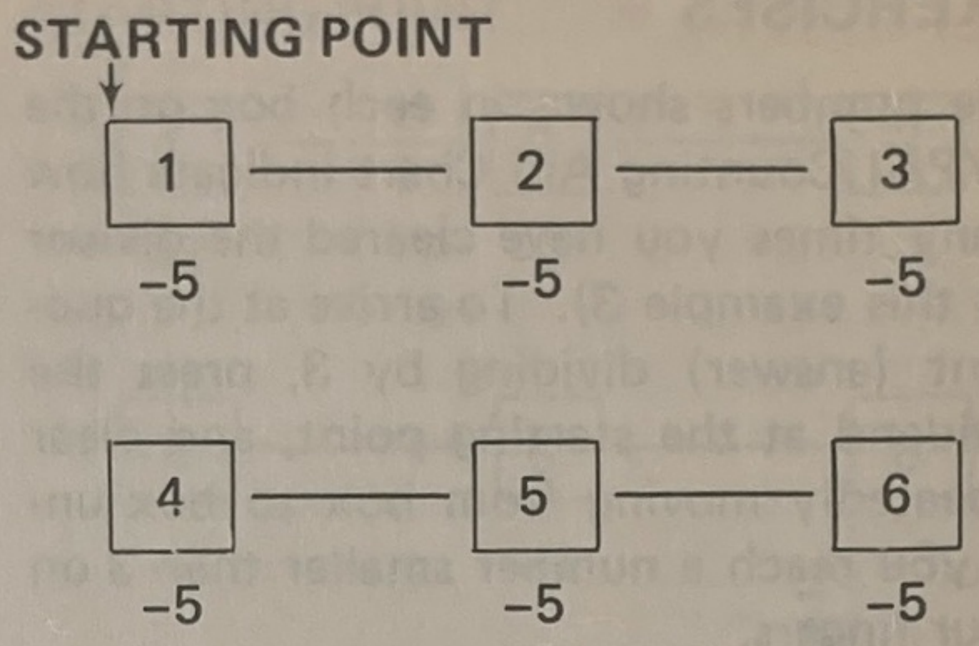
## EXERCISES

The numbers shown in each box on the HYPAI Counting Aid Chart indicate how many times you have cleared the divisor (in this example 4). To arrive at the quotient (answer) dividing by 4, press the dividend at the starting point, and clear repeatedly moving from box to box until you reach a number smaller than 4 on your fingers.

| Dividend ÷ Divisor | Dividend ÷ Divisor |
|---|---|
| 8 ÷ 4 = □ | 4 ÷ 4 = □ |
| 16 ÷ 4 = □ | 12 ÷ 4 = □ |
| 24 ÷ 4 = □ | 20 ÷ 4 = □ |
| 32 ÷ 4 = □ | 28 ÷ 4 = □ |
| 0 ÷ 4 = □ | 36 ÷ 4 = □ |
| | |
| 28 ÷ 4 = □ | 24 ÷ 4 = □ |
| 20 ÷ 4 = □ | 16 ÷ 4 = □ |
| 36 ÷ 4 = □ | 0 ÷ 4 = □ |
| 4 ÷ 4 = □ | 8 ÷ 4 = □ |
| 12 ÷ 4 = □ | 32 ÷ 4 = □ |
| | |
| 5 ÷ 4 = □ | 9 ÷ 4 = □ |
| 13 ÷ 4 = □ | 17 ÷ 4 = □ |
| 21 ÷ 4 = □ | 25 ÷ 4 = □ |
| 29 ÷ 4 = □ | 33 ÷ 4 = □ |
| 33 ÷ 4 = □ | 1 ÷ 4 = □ |

## FIVES

Before using this HYPAI Counting Aid Chart, the student must practice this exercise orally. Clear fives, one-by-one, from fifty saying *one two three four five, one two three four five,*...etc. until zero is reached.




EXAMPLE: 25÷5

You will be subtracting five five times. The chart enables you to keep count of the number of times five is being subtracted since the number in the boxes indicates that number. The answer is always found by looking at the number in the box you reached, with the remainder showing on the fingers.

*Procedure:*

Move your fingers from square to square as you say, "Clear five...*one two three four five, one two three four five,*...etc." When you reach square 5, read the number in the box where you stopped. After this method is practiced many times, you will be able to move only your eyes from square to square as you manipulate your fingers. Ultimately, you are able to keep track mentally of how many times you have subtracted a number.

## EXERCISES

The numbers shown in each box on the HYPAI Counting Aid Chart indicate how many times you have cleared the divisor (in this example 5). To arrive at the quotient (answer) dividing by 5, press the dividend at the starting point, and clear repeatedly moving from box to box until you reach a number smaller than 5 on your fingers.

| **Dividend ÷ Divisor** | **Dividend ÷ Divisor** |
|---|---|
| 10 ÷ 5 = ☐ | 5 ÷ 5 = ☐ |
| 20 ÷ 5 = ☐ | 15 ÷ 5 = ☐ |
| 30 ÷ 5 = ☐ | 25 ÷ 5 = ☐ |
| 40 ÷ 5 = ☐ | 35 ÷ 5 = ☐ |
| 0 ÷ 5 = ☐ | 45 ÷ 5 = ☐ |
| 35 ÷ 5 = ☐ | 30 ÷ 5 = ☐ |
| 25 ÷ 5 = ☐ | 20 ÷ 5 = ☐ |
| 45 ÷ 5 = ☐ | 0 ÷ 5 = ☐ |
| 5 ÷ 5 = ☐ | 10 ÷ 5 = ☐ |
| 15 ÷ 5 = ☐ | 40 ÷ 5 = ☐ |
| 7 ÷ 5 = ☐ | 12 ÷ 5 = ☐ |
| 17 ÷ 5 = ☐ | 28 ÷ 5 = ☐ |
| 24 ÷ 5 = ☐ | 36 ÷ 5 = ☐ |
| 39 ÷ 5 = ☐ | 42 ÷ 5 = ☐ |
| 47 ÷ 5 = ☐ | 2 ÷ 5 = ☐ |

At this point, it should be clear how to use the Counting Aid Chart. One final point in division should be made. Up to now, no problems have been presented in which the answer would be greater than nine. However, as long as the divisor is a single digit, there is no problem in extending the theory. All one has to remember are the following rules:

1) Approach the dividend column-by-column, left to right.

2) After clearing the divisor as much as possible, note the remainder on the right hand. This remainder must be transferred to the left hand before starting the next column.

EXAMPLE: 564 ÷ 4

STEP 1 5 ÷ 4 = 1 R 1
STEP 2 16 ÷ 4 = 4
STEP 3 4 ÷ 4 = 1

564 ÷ 4 = 141

## CONCLUSION

Now that you have completed the stages of addition, multiplication, subtraction and division, you are aware that the Chisanbop™ Method is not just a series of fingering exercises. The author has examined many of the conventional procedures to suggest a simpler, more consistent rule where possible. The use of the fingers is important since it provides one more way to reach the brain. The use of the fingers is just one small part of the Chisanbop™ Method.

This book presents the most up-to-date exposition of Chisanbop™ Theory. It contains over a dozen procedures developed in the last two years. The author continues to teach young students and searches for ways to present mathematics to them in an easy, enjoyable way. Any of his new insights will be presented periodically in revised versions of this manual.

Chisanbop™ Theory flows from stage to stage. Ideally it should be learned starting with the 1st Stage and then followed by the

2nd, 3rd, and 4th Stages. However, some teachers will find their older and brighter students impatient. With these students teachers should start with Stage 3, and only give instruction on the earlier stages when the students are unable to perform some 3rd Stage operation. As a general rule, however, this should not be done unless the students are at least in the 4th grade and have memorized most of their number facts.

The sample problems presented in this book are mostly illustrative material. Teachers can find many more problems presented in sequence in the students' workbooks which are published by the American Book Company. Teachers wanting even more insight into this method should avail themselves of the opportunity of attending one of the Chisanbop™ Workshops now being held throughout the country.

The school districts that have had the Chisanbop™ Method in use have viewed it as a mathematical breakthrough that enables children to expand their mathematical abilities at a faster pace than ever before.

We trust that you have found this program fun and rewarding, and hope that you will be one of the thousands who will continue to learn Chisanbop™ with this book. We believe that to do so will give you a life-long advantage in the understanding and utilization of mathematics.

It is the author's highest hope and greatest sense of satisfaction that his work is making a contribution to the students of the world, enabling them to study mathematics without pain.

The reason for this required delay in the introduction to subtraction is that the finger manipulations for subtraction are the exact opposite of those required for addition. Thus, if a student attempts to learn the subtraction manipulations before being completely secure with the addition maneuvers, he will become confused with the two operations, and his progress will be slowed. Delaying the study of subtraction, will, in the long run, actually help to decrease the total time needed to learn Chisanbop™.

There is one difference in the presentation of subtraction. Subtraction Stage 1 is presented in two parts, A and B. Part A introduces the student into the concept of subtraction but does not present any problems where borrowing would be required. Since borrowing is such a difficult topic, and since the author has developed a unique method, HYPAI Trading, for inculcating the concept, a separate chapter is reserved.

Chisanbop™ is also appropriate for children with various learning capacities. The young and slow students are grateful for the new found accuracy. The older brighter students are excited by the speed which Chisanbop™ can bring to their calculations.

Now that you have some idea about the nature of Chisanbop™, we recommend that you take each instruction carefully. Even though you might not understand why a certain instruction is made, each instruction is part of the methodology. By the time you have finished this manual, you will see how Chisanbop™ really does give insight into the nature of numbers, the decimal system, and the arithmetical operations of addition, multiplication and missing addends. However, those insights come only from following each part of the method as outlined in this manual.

$12.95

As millions of TV viewers watched, seven-year-old children pitted their math skills against adults on national television. In test after test of speed and accuracy, the children won. What is more amazing, they used only their fingers to perform complex mathematical computations; the adults worked with hand calculators. These children had learned Chisanbop, a finger calculation method of addition, subtraction, multiplication, and division.

This unique system—now widely accepted by educators and used in school systems nationwide to supplement traditional mathematics instruction—is presented here. The system is explained step-by-step; exercises for developing skill and speed accompany each step.

The speed and accuracy of the system will soon promote self-confidence in your children and make mathematics an exciting adventure.

Chisanbop was created in Korea thirty years ago by Sung Jin Pai. His son, Hang Young Pai, simplified and refined the system, and brought Chisanbop to this country little more than three years ago. Since then, Hang Young Pai has been active nationwide in teaching young children and teachers alike this unique system.

**Hang Young Pai** was born in Korea in 1943, the son of Sung Jin Pai, a mathematician who had devised a finger calculation method—CHI SAN BOP—for use by advanced math students to achieve very rapid calculation in test situations. The young Mr. Pai learned the method and when, several years after earning his degree in business administration from the University of Korea, he emigrated to the United States, he began to teach a simplified version to American children. Hang Young Pai refined, developed, and published a complete instruction and practice program oriented to the American school curriculum and has trained 5,000 teachers through workshops to instruct children in the classrooms of the U.S., Canada, and many other countries (including Saudi Arabia) in the use of this calculation method. He continues to train teachers and to lecture on Chisanbop™ at colleges and universities and before audiences at statewide and national conventions dealing with education. He continues to develop the concepts and skills of Chisanbop while also acting as President of Chisanbop Enterprises, Inc. Mr. Pai is married and the father of two children.

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